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Pre-calculus 11

NOVA SCOTIA

CURRICULUM GUIDE

Pre-calculus 11

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Introduction

Background and Rationale

Mathematics curriculum is shaped by a vision that fosters the development of mathematically literate students who can extend and apply their learning and who are effective participants in society. It is essential that the mathematics curriculum reflect current research in mathematics instruction. To achieve this goal, Western and Northern Canadian Protocol's (WNCP) *The Common Curriculum Framework for Grades 10–12 Mathematics* (2008) has been adopted as the basis for the new mathematics curriculum in Nova Scotia.

The Common Curriculum Framework was developed by the seven ministries of education (Alberta, British Columbia, Manitoba, Northwest Territories, Nunavut, Saskatchewan, and Yukon Territory) in collaboration with teachers, administrators, parents, business representatives, post-secondary educators, and others. The framework identifies beliefs about mathematics, general and specific student outcomes, and performance indicators agreed upon by the seven jurisdictions. The outcomes and performance indicators have been adapted for Nova Scotia. This document is based on both national and international research by the WNCP and the National Council of Teachers of Mathematics (NCTM).

There is an emphasis in the Nova Scotia curriculum on particular key concepts at each grade that will result in greater depth of understanding and, ultimately, stronger student achievement. There is also a greater emphasis on number sense and operations concepts in the early grades to ensure students develop a solid foundation in numeracy.

Purpose

This document provides sets of outcomes and performance indicators to be used as a mandated common base for defining mathematics curriculum expectations. This common base should result in consistent student outcomes in mathematics within the province of Nova Scotia. It should also enable easier transfer for students moving within the province or from any jurisdiction that has adopted the WNCP framework. This document is intended to clearly communicate to all education partners across the province the high expectations for students' mathematical learning.

Program Design and Components

Pathways

The *Common Curriculum Framework for Grades 10–12 Mathematics* (WNCP 2008), on which the Nova Scotia Mathematics 10–12 curriculum is based, includes pathways and topics rather than strands as in *The Common Curriculum Framework for K–9 Mathematics* (WNCP 2006). In Nova Scotia, four pathways are available: Mathematics Essentials, Mathematics at Work, Mathematics, and Pre-calculus.

Each topic area requires that students develop a conceptual knowledge base and skill set that will be useful to whatever pathway they have chosen. The topics covered within a pathway are meant to build upon previous knowledge and to progress from simple to more complex conceptual understandings.

Goals of Pathways

The goals of all four pathways are to provide prerequisite attitudes, knowledge, skills, and understandings for specific post-secondary programs or direct entry into the work force. All four pathways provide students with mathematical understandings and critical-thinking skills. It is the choice of topics through which those understandings and skills are developed that varies among pathways. When choosing a pathway, students should consider their interests, both current and future. Students, parents, and educators are encouraged to research the admission requirements for post-secondary programs of study as they vary by institution and by year.

Design of Pathways

Each pathway is designed to provide students with the mathematical understandings, rigour, and critical-thinking skills that have been identified for specific post-secondary programs of study and for direct entry into the work force.

The content of the Mathematics Essentials courses was designed in Nova Scotia to fill a specific need for Nova Scotia students. The content of each of the Mathematics at Work, Mathematics, and Pre-calculus pathways has been based on the *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings* (Alberta Education 2006) and on consultations with mathematics teachers.

MATHEMATICS ESSENTIALS (GRADUATION)

This pathway is designed to provide students with the development of the skills and understandings required in the workplace, as well as those required for everyday life at home and in the community. Students will become better equipped to deal with mathematics in the real world and will become more confident in their mathematical abilities.

MATHEMATICS AT WORK (GRADUATION)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into some college programs and for direct entry into the work force. Topics include financial mathematics, algebra, geometry, measurement, number, and statistics and probability.

MATHEMATICS (ACADEMIC)

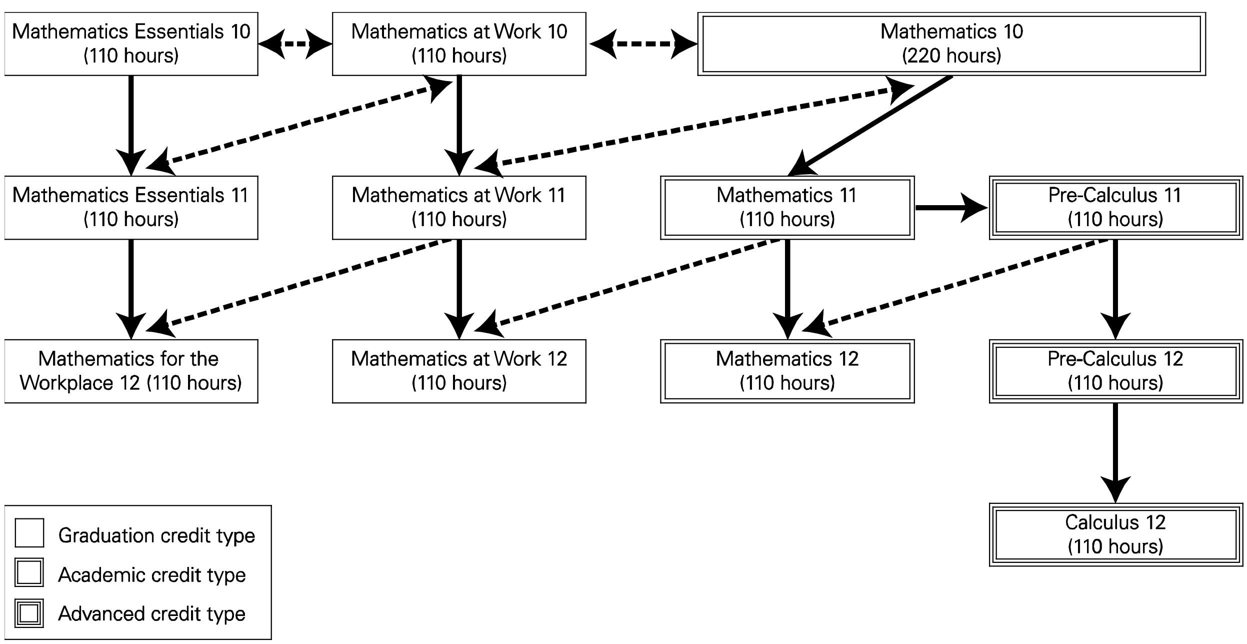
This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for post-secondary studies in programs that require an academic or pre-calculus mathematics credit. Topics include financial mathematics, geometry, measurement, number, logical reasoning, relations and functions, and statistics and probability. **Note:** After completion of Mathematics 11, students have the choice of an academic or pre-calculus pathway.

PRE-CALCULUS (ADVANCED)

This pathway is designed to provide students with the mathematical understandings and critical-thinking skills identified for entry into post-secondary programs that require the study of theoretical calculus. Topics include algebra and number, measurement, relations and functions, trigonometry, and permutations, combinations, and binomial theorem.

Pathways and Courses

The graphic below summarizes the pathways and courses offered.



Instructional Focus

Each pathway in senior high mathematics pathways is arranged by topics. Students should be engaged in making connections among concepts both within and across topics to make mathematical learning experiences meaningful.

Teachers should consider the following points when planning for instruction and assessment.

- The mathematical processes that are identified with the outcome are intended to help teachers select effective pedagogical approaches for the teaching and learning of the outcome.
- All seven mathematical processes must be integrated throughout teaching and learning approaches and should support the intent of the outcomes.
- Wherever possible, meaningful contexts should be used in examples, problems, and projects.
- Instruction should flow from simple to complex and from concrete to abstract.
- The assessment plan for the course should be a balance of assessment for learning and assessment of learning.

The focus of student learning should be on developing a conceptual and procedural understanding of mathematics. Students' conceptual understanding and procedural understanding must be directly related.

Assessment

Ongoing assessment for learning is essential to effective teaching and learning. Research has shown that assessment for learning (formative assessment) practices produce significant and often substantial learning gains, close achievement gaps, and build students' ability to learn new skills (Black & Wiliam 1998; OECD 2006). Student involvement in assessment promotes learning. Timely and effective teacher feedback and student self-assessment allow students to reflect on and articulate their understanding of mathematical concepts and ideas.

Assessment in the classroom includes

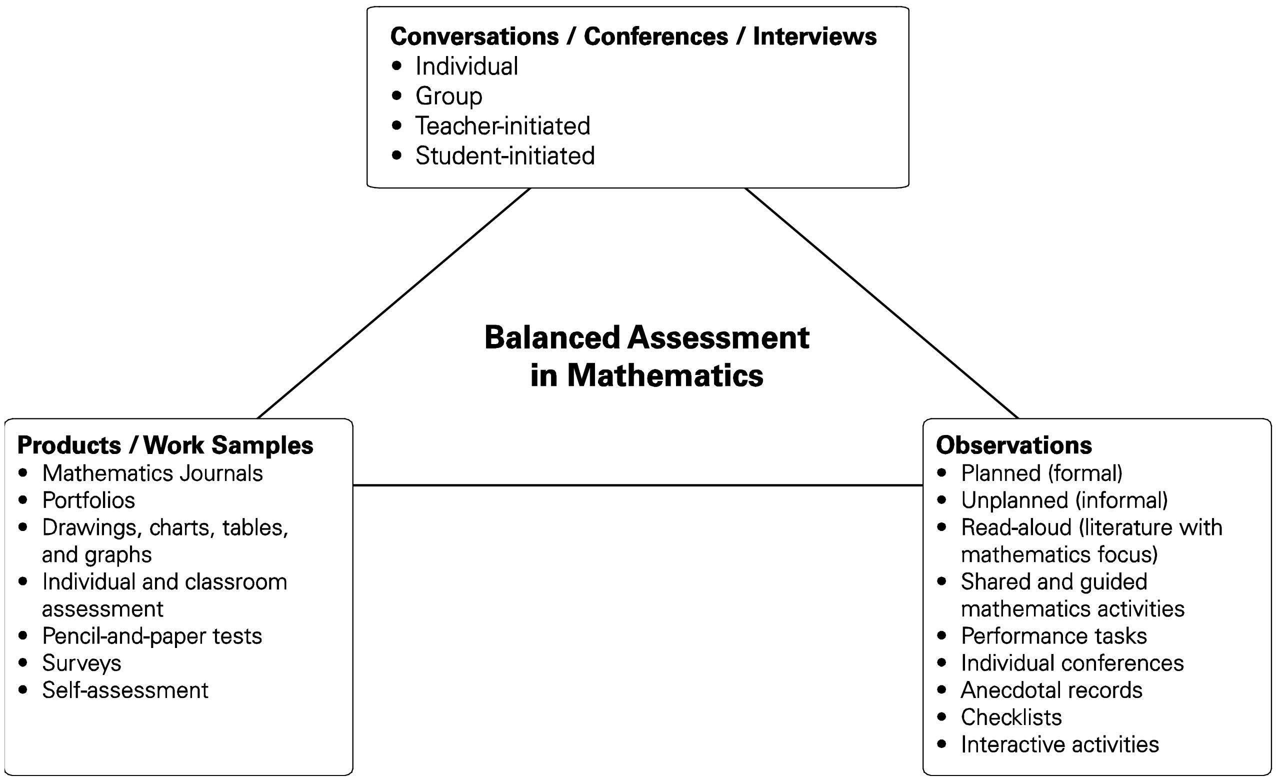
- providing clear goals, targets, and learning outcomes
- using exemplars, rubrics, and models to help clarify outcomes and identify important features of the work
- monitoring progress towards outcomes and providing feedback as necessary
- encouraging self-assessment
- fostering a classroom environment where conversations about learning take place, where students can check their thinking and performance and develop a deeper understanding of their learning

(Davies 2000)

Assessment for learning practices act as the scaffolding for learning, which only then can be measured through assessment of learning (summative assessment). Assessment of learning tracks student progress, informs instructional programming, and aids in decision making. Both forms of assessment are necessary to guide teaching, stimulate learning, and produce achievement gains.

Assessment of student learning should

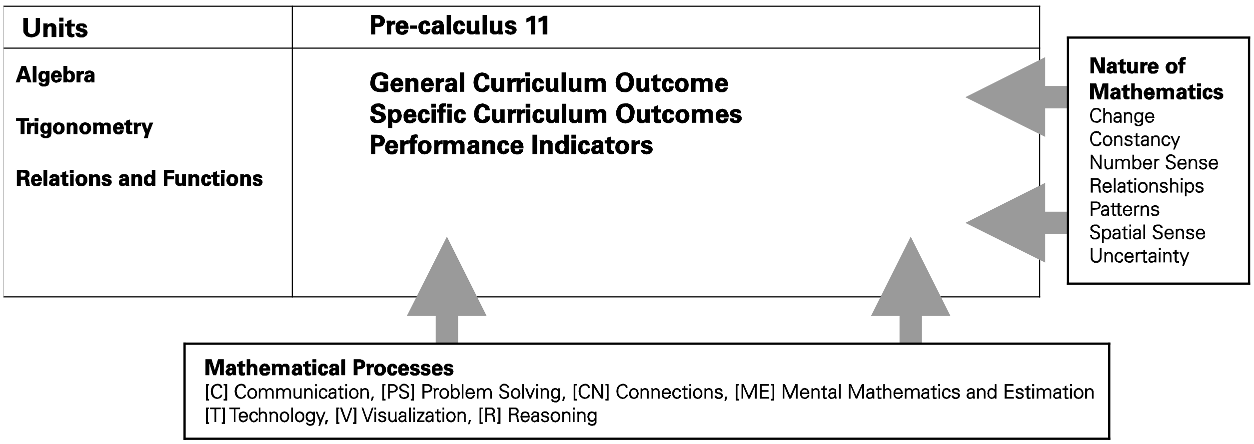
- align with curriculum outcomes
- clearly define criteria for success
- make explicit the expectations for students' performance
- use a wide variety of assessment strategies and tools
- yield useful information to inform instruction



Outcomes

Conceptual Framework for Mathematics 10–12

The chart below provides an overview of how mathematical processes and the nature of mathematics influence learning outcomes.



(Adapted with permission from Western and Northern Canadian Protocol, *The Common Curriculum Framework for K–9 Mathematics*, p. 5. All rights reserved.)

Structure of the Pre-calculus 11 Curriculum

Units

Pre-calculus 11 comprises three units:
- Algebra and Number (AN) (30–35 hours)
- Trigonometry (T) (10 hours)
- Relations and Functions (RF) (60–70 hours)

Outcomes and Performance Indicators

The Nova Scotia curriculum is stated in terms of general curriculum outcomes, specific curriculum outcomes, and performance indicators.

General Curriculum Outcomes (GCOs)

General curriculum outcomes are overarching statements about what students are expected to learn in each strand/sub-strand. The GCO for each strand/sub-strand is the same throughout the pathway.

Algebra and Number (AN)

Students will be expected to develop algebraic reasoning and number sense.

Trigonometry (T)

Students will be expected to develop trigonometric reasoning.

Relations and Functions (RF)

Students will be expected to develop algebraic and graphical reasoning through the study of relations.

Specific Curriculum Outcomes (SCOs) and Performance Indicators

Specific curriculum outcomes are statements that identify specific concepts and related skills underpinned by the understanding and knowledge attained by students as expected for a given grade.

Performance indicators are samples of how students may demonstrate their performance of the goals of a specific curriculum outcome. The range of samples provided is meant to reflect the scope of the SCO. In the SCOs, the word **including** indicates that any ensuing items *must* be addressed to fully achieve the learning outcome. The phrase **such as** indicates that the ensuing items are provided for clarification only and are *not* requirements that must be addressed to fully achieve the learning outcome. The word **and** used in an outcome indicates that both ideas must be addressed to achieve the learning outcome, although not necessarily at the same time or in the same question.

ALGEBRA AND NUMBER (AN)

AN01 Students will be expected to demonstrate an understanding of the absolute value of real numbers.

Performance Indicators

AN01.01 Determine the distance of two real numbers of the form $\pm a$, $a \in R$, from 0 on a number line, and relate this to the absolute value of $a(|a|)$.
AN01.02 Determine the absolute value of a positive or negative real number.
AN01.03 Explain, using examples, how distance between two points on a number line can be expressed in terms of absolute value.
AN01.04 Determine the absolute value of a numerical expression.
AN01.05 Compare and order the absolute values of real numbers in a given set.

AN02 Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.

Performance Indicators

AN02.01 Compare and order radical expressions with numerical radicands in a given set.
AN02.02 Express an entire radical with a numerical radicand as a mixed radical.
AN02.03 Express a mixed radical with a numerical radicand as an entire radical.
AN02.04 Perform one or more operations to simplify radical expressions with numerical or variable radicands.
AN02.05 Rationalize the denominator of a radical expression with monomial or binomial denominators.
AN02.06 Describe the relationship between rationalizing a binomial denominator of a rational expression and the product of the factors of a difference of squares expression.
AN02.07 Explain, using examples, that $(-x)^2 = x^2$, $\sqrt{x^2} = |x|$, and $\sqrt{x^2} \neq \pm x$.
AN02.08 Identify the values of the variable for which a given radical expression is defined.
AN02.09 Solve a problem that involves radical expressions.

AN03 Students will be expected to solve problems that involve radical equations (limited to square roots).

Performance Indicators

(It is intended that the equations will have no more than two radicals.)

AN03.01 Determine any restrictions on values for the variable in a radical equation.

AN03.02 Determine the roots of a radical equation algebraically, and explain the process used to solve the equation.

AN03.03 Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation.

AN03.04 Explain why some roots determined in solving a radical equation algebraically are extraneous.

AN03.05 Solve problems by modelling a situation using a radical equation.

AN04 Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials).

Performance Indicators

AN04.01 Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.

AN04.02 Explain why a given value is non-permissible for a given rational expression.

AN04.03 Determine the non-permissible values for a rational expression.

AN04.04 Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.

AN04.05 Simplify a rational expression.

AN04.06 Explain why the non-permissible values of a given rational expression and its simplified form are the same.

AN04.07 Identify and correct errors in a simplification of a rational expression, and explain the reasoning.

AN05 Students will be expected to perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials).

Performance Indicators

AN05.01 Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.

AN05.02 Determine the non-permissible values when performing operations on rational expressions.

AN05.03 Determine, in simplified form, the sum or difference of rational expressions with the same denominator.

AN05.04 Determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors.

AN05.05 Determine, in simplified form, the product or quotient of rational expressions.

AN05.06 Simplify an expression that involves two or more operations on rational expressions.

AN06 Students will be expected to solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials, or trinomials).

Performance Indicators

AN06.01 Determine the non-permissible values for the variable in a rational equation.
AN06.02 Determine the solution to a rational equation algebraically, and explain the process used to solve the equation.
AN06.03 Explain why a value obtained in solving a rational equation may not be a solution of the equation.
AN06.04 Solve problems by modelling a situation using a rational equation.

TRIGONOMETRY (T)

T01 Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°).

Performance Indicators

T01.01 Sketch an angle in standard position, given the measure of the angle.
T01.02 Determine the reference angle for an angle in standard position.
T01.03 Explain, using examples, how to determine the angles from 0° to 360° that have the same reference angle as a given angle.
T01.04 Illustrate, using examples, that any angle from 90° to 360° is the reflection in the x-axis and/or the y-axis of its reference angle.
T01.05 Determine the quadrant in which a given angle in standard position terminates.
T01.06 Draw an angle in standard position given any point $P(x, y)$ on the terminal arm of the angle.
T01.07 Illustrate, using examples, that the points $P(x, y)$, $P(-x, y)$, $P(-x, -y)$, and $P(x, -y)$ are points on the terminal sides of angles in standard position that have the same reference angle.

T02 Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.

Performance Indicators

T02.01 Determine, using the Pythagorean theorem or the distance formula, the distance from the origin to a point $P(x, y)$ on the terminal arm of an angle.
T02.02 Determine the value of $\sin\theta$, $\cos\theta$, or $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ.
T02.03 Determine, without the use of technology, the value of $\sin\theta$, $\cos\theta$, or $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ, where θ = 0°, 90°, 180°, 270°, or 360°.
T02.04 Determine the sign of a given trigonometric ratio for a given angle, without the use of technology, and explain.
T02.05 Solve, for all values of θ, an equation of the form $\sin\theta = a$ or $\cos\theta = a$, where $-1 \le a \le 1$, and an equation of the form $\tan\theta = a$, where a is a real number.
T02.06 Determine the exact value of the sine, cosine, or tangent of a given angle with a reference angle of 30°, 45°, or 60°.
T02.07 Describe patterns in and among the values of the sine, cosine, and tangent ratios for angles from 0° to 360°.
T02.08 Sketch a diagram to represent a problem.
T02.09 Solve a contextual problem, using trigonometric ratios.

T03 Students will be expected to solve problems, using the cosine law and sine law, including the ambiguous case.

Performance Indicators

T03.01 Sketch a diagram to represent a problem that involves a triangle without a right angle.
T03.02 Solve, using primary trigonometric ratios, a triangle that is not a right triangle.
T03.03 Explain the steps in a given proof of the sine law or cosine law.
T03.04 Sketch a diagram and solve a problem, using the cosine law.
T03.05 Sketch a diagram and solve a problem, using the sine law.
T03.06 Describe and explain situations in which a problem may have no solution, one solution, or two solutions.

RELATIONS AND FUNCTIONS (RF)

RF01 Students will be expected to factor polynomial expressions of the following form where *a*, *b*, and *c* are rational numbers.

- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
- $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$
- $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$

Performance Indicators

RF01.01 Factor a given polynomial expression that requires the identification of common factors.
RF01.02 Determine whether a given binomial is a factor for a given polynomial expression, and explain why or why not.
RF01.03 Factor a given polynomial expression of the form
- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$

RF01.04 Factor a given polynomial expression that has a quadratic pattern, including
- $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$
- $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$

RF02 Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.

Performance Indicators

RF02.01 Create a table of values for $y = |f(x)|$, given a table of values for $y = f(x)$.
RF02.02 Generalize a rule for writing absolute value functions in piecewise notation.
RF02.03 Sketch the graph of $y = |f(x)|$; state the intercepts, domain, and range; and explain the strategy used.
RF02.04 Solve an absolute value equation graphically, with or without technology.
RF02.05 Solve, algebraically, an equation with a single absolute value, and verify the solution.
RF02.06 Explain why the absolute value equation $|f(x)| < 0$ has no solution.
RF02.07 Determine and correct errors in a solution to an absolute value equation.
RF02.08 Solve a problem that involves an absolute value function.

RF03 Students will be expected to analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, x-intercept, and y-intercept.

(*This outcome will focus on quadratic functions written in the vertex form,* $f(x) = a(x - h)^2 + k$ *. Note that some of the performance indicators for this outcome have been greyed out. This is because they were addressed in Mathematics 11, and it is the intent of this course to extend and deepen student understanding of these performance indicators.*)

Performance Indicators

RF03.01 Explain why a function given in the form $y = a(x - p)^2 + q$ is a quadratic function.

RF03.02 Compare the graphs of a set of functions of the form $y = ax^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of a.

RF03.03 Compare the graphs of a set of functions of the form $y = x^2 + q$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of q.

RF03.04 Compare the graphs of a set of functions of the form $y = (x - p)^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of p.

RF03.05 Determine the coordinates of the vertex for a quadratic function of the form, $y = a(x - p)^2 + q$ and verify with or without technology.

RF03.06 Generalize, using inductive reasoning, a rule for determining the coordinates of the vertex for quadratic functions of the form $y = a(x - p)^2 + q$.

RF03.07 Sketch the graph of $y = a(x - p)^2 + q$, using transformations, and identify the vertex, domain and range, direction of opening, axis of symmetry, and x- and y-intercepts.

RF03.08 Explain, using examples, how the values of a and q may be used to determine whether a quadratic function has zero, one, or two x-intercepts.

RF03.09 Write a quadratic function in the form $y = a(x - p)^2 + q$ for a given graph or a set of characteristics of a graph.

RF04 Students will be expected to analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including vertex, domain and range, direction of opening, axis of symmetry, x-intercept and y-intercept, and to solve problems.

Performance Indicators

RF04.01 Explain the reasoning for the process of completing the square as shown in a given example.

RF04.02 Write a quadratic function given in the form $y = ax^2 + bx + c$ as a quadratic function in the form $y = a(x - p)^2 + q$ by completing the square.

RF04.03 Identify, explain, and correct errors in an example of completing the square.

RF04.04 Determine the characteristics of a quadratic function given in the form $y = ax^2 + bx + c$, and explain the strategy used.

RF04.05 Sketch the graph of a quadratic function given in the form $y = ax^2 + bx + c$.

RF04.06 Verify, with or without technology, that a quadratic function in the form $y = ax^2 + bx + c$ represents the same function as a given quadratic function in the form $y = a(x - p)^2 + q$.

RF04.07 Write a quadratic function that models a given situation, and explain any assumptions made.

RF04.08 Solve a problem, with or without technology, by analyzing a quadratic function.

RF05 Students will be expected to solve problems that involve quadratic equations.

Performance Indicators

RF05.01 Explain, using examples, the relationship among the roots of a quadratic equation, the zeros of the corresponding quadratic function, and the *x*-intercepts of the graph of the quadratic function.

RF05.02 Derive the quadratic formula, using deductive reasoning.

RF05.03 Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by using strategies such as

- determining square roots
- factoring
- completing the square
- applying the quadratic formula
- graphing its corresponding function

RF05.04 Select a method for solving a quadratic equation, justify the choice, and verify the solution.

RF05.05 Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real roots, and relate the number of zeros to the graph of the corresponding quadratic function.

RF05.06 Identify and correct errors in a solution to a quadratic equation.

RF05.07 Solve a problem by

- analyzing a quadratic equation
- determining and analyzing a quadratic equation

RF06 Students will be expected to solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.

Performance Indicators

(It is intended that the quadratic equations be limited to those that correspond to quadratic functions.)

RF06.01 Model a situation, using a system of linear-quadratic or quadratic-quadratic equations.

RF06.02 Relate a system of linear-quadratic or quadratic-quadratic equations to the context of a given problem.

RF06.03 Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations graphically, with technology.

RF06.04 Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations algebraically.

RF06.05 Explain the meaning of the points of intersection of a system of linear-quadratic or quadratic-quadratic equations.

RF06.06 Explain, using examples, why a system of linear-quadratic or quadratic-quadratic equations may have zero, one, two, or an infinite number of solutions.

RF06.07 Solve a problem that involves a system of linear-quadratic or quadratic-quadratic equations, and explain the strategy used.

RF07 Students will be expected to solve problems that involve linear and quadratic inequalities in two variables.

Performance Indicators

RF07.01 Explain, using examples, how test points can be used to determine the solution region that satisfies an inequality.
RF07.02 Explain, using examples, when a solid or broken line should be used in the solution for an inequality.
RF07.03 Sketch, with or without technology, the graph of a linear or quadratic inequality.
RF07.04 Solve a problem that involves a linear or quadratic inequality.

RF08 Students will be expected to solve problems that involve quadratic inequalities in one variable.

Performance Indicators

RF08.01 Determine the solution of a quadratic inequality in one variable, using strategies such as case analysis, graphing, roots and test points, or sign analysis; and explain the strategy used.
RF08.02 Represent and solve a problem that involves a quadratic inequality in one variable.
RF08.03 Interpret the solution to a problem that involves a quadratic inequality in one variable.

RF09 Students will be expected to analyze arithmetic sequences and series to solve problems.

Performance Indicators

RF09.01 Identify the assumption(s) made when defining an arithmetic sequence or series.
RF09.02 Provide and justify an example of an arithmetic sequence.
RF09.03 Derive a rule for determining the general term of an arithmetic sequence.
RF09.04 Describe the relationship between arithmetic sequences and linear functions.
RF09.05 Determine t_1, d, n, or tn in a problem that involves an arithmetic sequence.
RF09.06 Derive a rule for determining the sum of n terms of an arithmetic series.
RF09.07 Determine t_1, d, n, or Sn in a problem that involves an arithmetic series.
RF09.08 Solve a problem that involves an arithmetic sequence or series.

RF10 Students will be expected to analyze geometric sequences and series to solve problems.

Performance Indicators

RF10.01 Identify assumptions made when identifying a geometric sequence or series.
RF10.02 Provide and justify an example of a geometric sequence.
RF10.03 Derive a rule for determining the general term of a geometric sequence.
RF10.04 Determine t_1, r, n, or tn in a problem that involves a geometric sequence.
RF10.05 Derive a rule for determining the sum of n terms of a geometric series.
RF10.06 Determine t_1, r, n, or Sn in a problem that involves a geometric series.
RF10.07 Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.
RF10.08 Explain why a geometric series is convergent or divergent.
RF10.09 Solve a problem that involves a geometric sequence or series.

RF11 Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).

Performance Indicators

RF11.01 Compare the graph of $y = \frac{1}{f(x)}$ to the graph of $y = f(x)$.

RF11.02 Identify, given a function $f(x)$, values of x for which $y = \frac{1}{f(x)}$ will have vertical asymptotes; and describe their relationship to the non-permissible values of the related rational expression.

RF11.03 Graph, with or without technology, $y = \frac{1}{f(x)}$, given $y = f(x)$ as a function or a graph, and explain the strategies used.

RF11.04 Graph, with or without technology, $y = f(x)$, given $y = \frac{1}{f(x)}$ as a function or a graph, and explain the strategies used.

Mathematical Processes

There are critical components that students must encounter in a mathematics program in order to achieve the goals of mathematics education and encourage lifelong learning in mathematics.

Students are expected to

- communicate in order to learn and express their understanding of mathematics (Communication [C])
- connect mathematical ideas to other concepts in mathematics, to everyday experiences, and to other disciplines (Connections [CN])
- demonstrate fluency with mental mathematics and estimation (Mental Mathematics and Estimation [ME])
- develop and apply new mathematical knowledge through problem solving (Problem Solving [PS])
- develop mathematical reasoning (Reasoning [R])
- select and use technologies as tools for learning and solving problems (Technology [T])
- develop visualization skills to assist in processing information, making connections, and solving problems (Visualization [V])

The Nova Scotia curriculum incorporates these seven interrelated mathematical processes that are intended to permeate teaching and learning. The key to these process standards is presented in a box, as shown below, with each specific outcome within the units.

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Communication [C]

Students need opportunities to read about, represent, view, write about, listen to, and discuss mathematical ideas. These opportunities allow students to create links between their own language and ideas, and the formal language and symbols of mathematics. Communication is important in clarifying, reinforcing, and modifying ideas, knowledge, attitudes, and beliefs about mathematics. Students should be encouraged to use a variety of forms of communication while learning mathematics.

Students also need to communicate their learning using mathematical terminology. Communication can help students make connections between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, written and symbolic—of mathematical ideas. Students must communicate *daily* about their mathematics learning. This enables them to reflect, to validate, and to clarify their thinking and provides teachers with insight into students' interpretations of mathematical meanings and ideas.

Emerging technologies enable students to engage in communication beyond the traditional classroom to gather data and share mathematical ideas.

Problem Solving [PS]

Problem solving is one of the key processes and foundations within the field of mathematics. Learning through problem solving should be the focus of mathematics at all grade levels. Students develop a true understanding of mathematical concepts and procedures when they solve problems in meaningful contexts.

When students encounter new situations and respond to questions of the type, How would you ... ? or How could you ... ?, the problem-solving approach is being modeled. Students develop their own problem-solving strategies by listening to, discussing, and trying different strategies.

In order for an activity to be problem-solving based, it must ask students to determine a way to get from what is known to what is sought. If students have already been given ways to solve the problem, it is not a problem, but practice. Students should not know the answer immediately. A true problem requires students to use prior learning in new ways and contexts. Problem solving requires and builds depth of conceptual understanding and student engagement. Students will be engaged if the problems relate to their lives, cultures, interests, families, or current events.

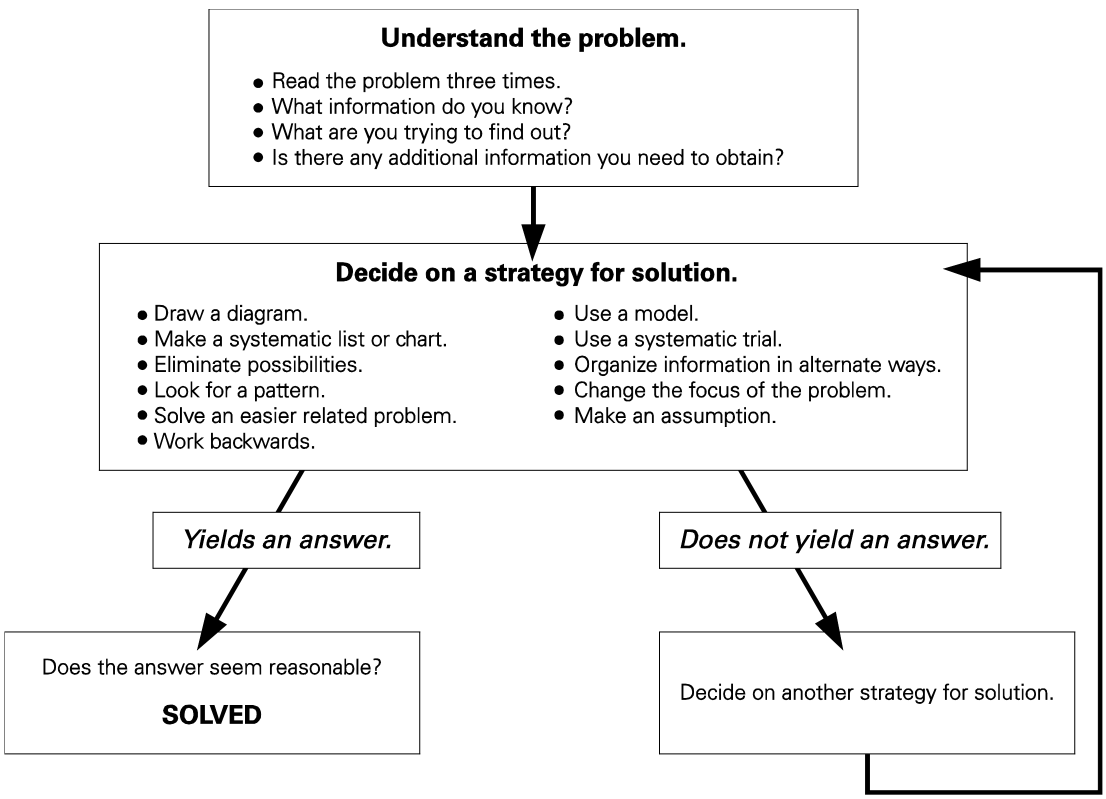
Both conceptual understanding and student engagement are fundamental in molding students' willingness to persevere in future problem-solving tasks. Problems are not just simple computations embedded in a story, nor are they contrived. They are tasks that are rich and open-ended, so there may be more than one way of arriving at a solution or there may be multiple answers. Good problems should allow for every student in the class to demonstrate their knowledge, skill, or understanding. Problem solving can vary from being an individual activity to a class (or beyond) undertaking.

In a mathematics class, there are two distinct types of problem solving: solving contextual problems outside of mathematics and solving mathematical problems. Finding the maximum profit given manufacturing constraints is an example of a contextual problem, while seeking and developing a general formula to solve a quadratic equation is an example of a mathematical problem.

Problem solving can also be considered in terms of engaging students in both inductive- and deductive-reasoning strategies. As students make sense of the problem, they will be creating conjectures and looking for patterns that they may be able to generalize. This part of the problem-solving process often involves inductive reasoning. As students use approaches to solving the problem, they often move into mathematical reasoning that is deductive in nature. It is crucial that students be encouraged to engage in both types of reasoning and be given the opportunity to consider the approaches and strategies used by others in solving similar problems.

Problem solving is a powerful teaching tool that fosters multiple, creative, and innovative solutions. Creating an environment where students openly look for, and engage in finding, a variety of strategies for solving problems empowers students to explore alternatives and develops confident, cognitive mathematical risk-takers.

A possible flow chart to share with students is as follows:



Connections [CN]

Contextualization and making connections to the experiences of learners are powerful processes in developing mathematical understanding. When mathematical ideas are connected to one another or to real-world phenomena, students can begin to view mathematics as useful, relevant, and integrated. Learning mathematics within contexts and making connections relevant to learners can validate past experiences and increase student willingness to participate and be actively engaged. The brain is constantly looking for and making connections.

"Because the learner is constantly searching for connections on many levels, educators need to *orchestrate the experiences* from which learners extract understanding. … *Brain research establishes and confirms that multiple complex and concrete experiences are essential for meaningful learning and teaching*." (Caine and Caine 1991, 5).

Mathematics should be viewed as an integrated whole rather than as the study of separate strands or units. Connections must also be made between and among the different representational modes—contextual, concrete, pictorial, linguistic/verbal, and symbolic. The process of making connections, in turn, facilitates learning. Concepts and skills should also be connected to everyday situations and other curricular areas. For example, when developing literacy skills students learn to make text-to-world, text-to-text, and text-to-self connections. Students can also make connections to make mathematics come alive through math-to-world, math-to-math, and math-to-self connections.

Mental Mathematics and Estimation [ME]

Mental mathematics is a combination of cognitive strategies that enhance flexible thinking and number sense. It is calculating mentally without the use of external aids.

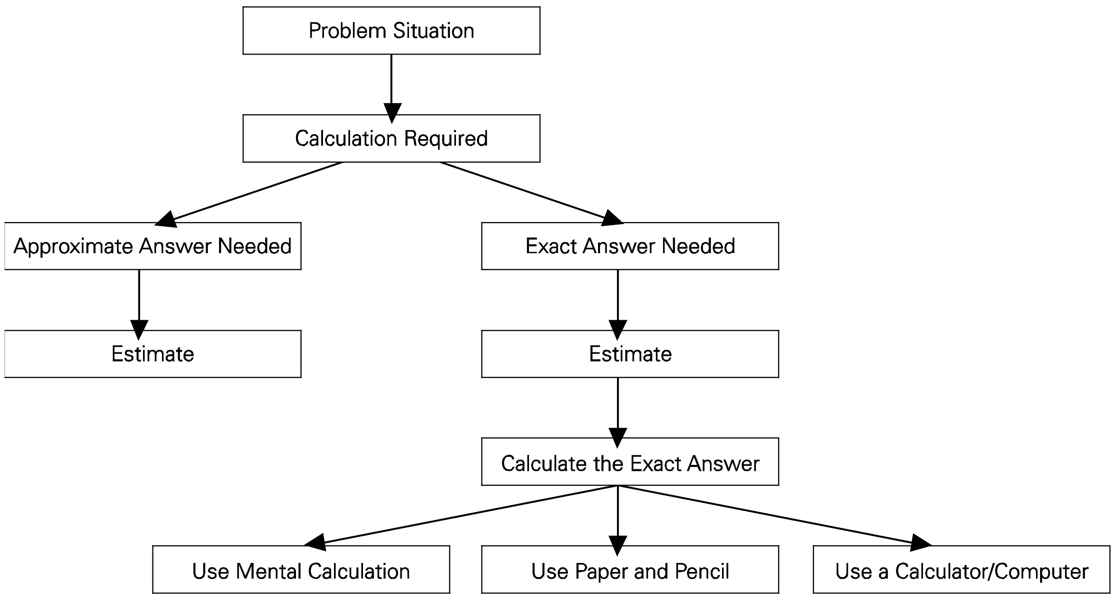
Mental mathematics enables students to determine answers without paper and pencil. It improves computational fluency by developing efficiency, accuracy and flexibility.

"Even more important than performing computational procedures or using calculators is the greater facility that students need—more than ever before—with estimation and mental math." (National Council of Teachers of Mathematics, May 2005).

Students proficient with mental mathematics "become liberated from calculator dependence, build confidence in doing mathematics, become more flexible thinkers, and are more able to use multiple approaches to problem solving." (Rubenstein 2001) Mental mathematics "provides a cornerstone for all estimation processes, offering a variety of alternative algorithms and nonstandard techniques for finding answers." (Hope 1988, v)

Estimation is a strategy for determining approximate values or quantities, usually by referring to benchmarks or using referents, or for determining the reasonableness of calculated values. Students need to know how, when, and what strategy to use when estimating. Estimation is used to make mathematical judgments and develop useful, efficient strategies for dealing with situations in daily life. When estimating, students need to learn which strategy to use and how to use it.

Students need to develop both mental mathematics and estimation skills through context and not in isolation so they are able to apply them to solve problems. Whenever a problem requires a calculation, students should follow the decision-making process as illustrated below.



The skill of estimation requires a sound knowledge of mental mathematics. Both are necessary to many everyday experiences, and students should be provided with frequent opportunities to practise these skills.

Technology [T]

Technology can be effectively used to contribute to and support the learning of a wide range of mathematical outcomes and enables students to explore and create patterns, examine relationships, test conjectures, and solve problems.

Calculators, computers, and other technologies can be used to

- explore and represent mathematical relationships and patterns in a variety of ways
- organize and display data
- extrapolate and interpolate
- assist with calculation procedures as part of solving problems
- increase the focus on conceptual understanding by decreasing the time spent on repetitive procedures
- reinforce the learning of foundational concepts
- develop personal procedures for mathematical operations
- simulate situations
- develop number and spatial sense
- generate and test inductive conjectures

Technology contributes to a learning environment in which the curiosity of students can lead to rich mathematical discoveries at all grade levels. The use of technology should not replace mathematical understanding. Instead, technology should be used as one of a variety of approaches and tools for creating mathematical understanding.

Visualization [V]

Visualization "involves thinking in pictures and images, and the ability to perceive, transform and recreate different aspects of the visual-spatial world." (Armstrong 1993, 10). The use of visualization in the study of mathematics provides students with opportunities to understand mathematical concepts and make connections among them. Visual images and visual reasoning are important components of number, spatial, and measurement sense. Number visualization occurs when students create mental representations of numbers. Being able to create, interpret and describe a visual representation is part of spatial sense and spatial reasoning. Spatial visualization and spatial reasoning enable students to describe the relationships among and between 3-D objects and 2-D shapes.

Measurement visualization goes beyond the acquisition of specific measurement skills. Measurement sense includes the ability to determine when to measure, when to estimate and to know several estimation strategies. (Shaw and Cliatt 1989, 150)

Visualization is fostered through the use of concrete materials, technology, and a variety of visual representations. It is through visualization that abstract concepts can be understood by the student. Visualization is a foundation to the development of abstract understanding, confidence, and fluency.

Reasoning [R]

Mathematical reasoning helps students think logically and make sense of mathematics. Students need to develop confidence in their abilities to reason and justify their mathematical thinking. Questions that challenge students to think, analyze, and synthesize help them develop an understanding of mathematics. All students need to be challenged to answer questions such as, Why do you believe that's true/correct? or What would happen if ...

Mathematical experiences provide opportunities for students to engage in inductive and deductive reasoning. Students use inductive reasoning when they explore and record results, analyze observations, make generalizations from patterns, and test these generalizations. Students use deductive reasoning when they reach new conclusions based upon the application of what is already known or assumed to be true.

Mathematics reasoning involves informal thinking, conjecturing, and validating—these help students understand that mathematics makes sense. Students are encouraged to justify, in a variety of ways, their solutions, thinking processes, and hypotheses. In fact, good reasoning is as important as finding correct answers. The thinking skills developed by focusing on reasoning can be used in daily life in a wide variety of contexts and disciplines.

Nature of Mathematics

Mathematics is one way of trying to understand, interpret, and describe our world. There are a number of components that define the nature of mathematics, and these are woven throughout this document. These components include change, constancy, number sense, relationships, patterns, spatial sense, and uncertainty.

Change

It is important for students to understand that mathematics is dynamic and not static. As a result, recognizing change is a key component in understanding and developing mathematics. Within mathematics, students encounter conditions of change and are required to search for explanations of that change. To make predictions, students need to describe and quantify their observations, look for patterns, and describe those quantities that remain fixed and those that change. For example, the sequence 4, 6, 8, 10, 12, … can be described as

- skip counting by 2s, starting from 4
- an arithmetic sequence, with first term 4 and a common difference of 2
- a linear function with a discrete domain

(Steen 1990, 184).

Students need to learn that new concepts of mathematics as well as changes to previously learned concepts arise from a need to describe and understand something new. Integers, decimals, fractions, irrational numbers, and complex numbers emerge as students engage in exploring new situations that cannot be effectively described or analyzed using whole numbers.

Students best experience change to their understanding of mathematical concepts as a result of mathematical play.

Constancy

Different aspects of constancy are described by the terms **stability**, **conservation**, **equilibrium**, **steady state** and **symmetry** (AAAS–Benchmarks 1993, 270). Many important properties in mathematics and science relate to properties that do not change when outside conditions change. Examples of constancy include the following:

- The area of a rectangular region is the same regardless of the methods used to determine the solution.
- The sum of the interior angles of any triangle is 180°.
- The theoretical probability of flipping a coin and getting heads is 0.5.
- Lines with constant slope.

Number Sense

Number sense, which can be thought of as intuition about numbers, is the most important foundation of numeracy. (British Columbia Ministry of Education, 2000, 146) Continuing to foster number sense is fundamental to growth of mathematical understanding.

A true sense of number goes well beyond the skills of simply counting, memorizing facts, and the situational rote use of algorithms. Students with strong number sense are able to judge the reasonableness of a solution, describe relationships between different types of numbers, compare quantities, and work with different representations of the same number to develop a deeper conceptual understanding of mathematics.

Number sense develops when students connect numbers to real-life experiences and use benchmarks and referents. This results in students who are computationally fluent, flexible with numbers, and have intuition about numbers. The evolving number sense typically comes as a by-product of learning rather than through direct instruction. However, number sense can be developed by providing rich mathematical tasks that allow students to make connections.

Relationships

Mathematics is used to describe and explain relationships. As part of the study of mathematics, students look for relationships among numbers, sets, shapes, objects, variables, and concepts. The search for possible relationships involves collecting and analyzing data, analyzing patterns and describing possible relationships visually, symbolically, orally or in written form.

Patterns

Mathematics is about recognizing, describing, and working with numerical and non-numerical patterns. Patterns exist in all of the mathematical topics, and it is through the study of patterns that students can make strong connections between concepts in the same and different topics.

Working with patterns also enables students to make connections beyond mathematics. The ability to analyze patterns contributes to how students understand their environment. Patterns may be represented in concrete, visual, auditory, or symbolic form. Students should develop fluency in moving from one representation to another.

Students need to learn to recognize, extend, create, and apply mathematical patterns. This understanding of patterns allows students to make predictions and justify their reasoning when solving problems. Learning to work with patterns helps develop students' algebraic thinking, which is foundational for working with more abstract mathematics.

Spatial Sense

Spatial sense involves the representation and manipulation of 3-D objects and 2-D shapes. It enables students to reason and interpret among 3-D and 2-D representations.

Spatial sense is developed through a variety of experiences with visual and concrete models. It offers a way to interpret and reflect on the physical environment and its 3-D or 2-D representations.

Some problems involve attaching numerals and appropriate units (measurement) to dimensions of objects. Spatial sense allows students to make predictions about the results of changing these dimensions.

Spatial sense is also critical in students' understanding of the relationship between the equations and graphs of functions and, ultimately, in understanding how both equations and graphs can be used to represent physical situations.

Uncertainty

In mathematics, interpretations of data and the predictions made from data may lack certainty. Events and experiments generate statistical data that can be used to make predictions. It is important to recognize that these predictions (interpolations and extrapolations) are based upon patterns that have a degree of uncertainty. The quality of the interpretation is directly related to the quality of the data. An awareness of uncertainty allows students to assess the reliability of data and data interpretation. Chance addresses the predictability of the occurrence of an outcome. As students develop their understanding of probability, the language of mathematics becomes more specific and describes the degree of uncertainty more accurately. This language must be used effectively and correctly to convey valuable messages.

Curriculum Document Format

This guide presents the mathematics curriculum so that a teacher may readily view the scope of the outcomes that students are expected to achieve during that year. Teachers are encouraged, however, to examine what comes before and what follows after, to better understand how students' learning at a particular grade level is part of a bigger picture of concept and skill development.

The order of presentation in no way assumes or prescribes a preferred order of presentation in the classroom, but simply lays out the specific curriculum outcomes in relation to the overarching general curriculum outcomes.

When a specific curriculum outcome is introduced, it is followed by the mathematical processes and performance indicators for that outcome. A scope and sequence is then provided, which relates the SCO to previous and next grade SCOs. Also for each SCO, there are background information, assessment strategies, suggested instructional strategies, and suggested models and manipulatives, mathematical vocabulary, and resource notes. For each section, the guiding questions should be used to help with unit and lesson preparation.

SCO

Mathematical Processes

[C] Communication	**[PS]** Problem Solving	**[CN]** Connections
[ME] Mental Mathematics and Estimation		
[T] Technology	**[V]** Visualization	**[R]** Reasoning

Performance Indicators

Describes observable indicators of whether students have met the specific outcome.

Scope and Sequence

Previous grade or course SCOs	Current course SCO	Following grade or course SCOs

Background

Describes the "big ideas" to be learned and how they relate to work in previous grade and work in subsequent courses.

Assessment, Teaching, and Learning

Assessment Strategies

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Sample tasks that can be used to determine students' prior knowledge.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Some suggestions for specific activities and questions that can be used for both instruction and assessment.

FOLLOW-UP ON ASSESSMENT

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?

Planning for Instruction

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcome and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?

SUGGESTED LEARNING TASKS

Suggestions for general approaches and strategies suggested for teaching this outcome.

Guiding Question

- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED MODELS AND MANIPULATIVES

MATHEMATICAL VOCABULARY

Resources/Notes

Contexts for Learning and Teaching

Beliefs about Students and Mathematics Learning

“Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge.” (National Council of Teachers of Mathematics 2000, 20).

- The Nova Scotia mathematics curriculum is based upon several key assumptions or beliefs about mathematics learning that have grown out of research and practice. These beliefs include the following:
- Mathematics learning is an active and constructive process.
- Learning is most effective when standards of expectation are made clear with ongoing assessment and feedback.
- Learners are individuals who bring a wide range of prior knowledge and experiences and who learn via various styles and at different rates.
- Leaning is most effective when standards of expectation are made clear with ongoing assessment and feedback.

Students are curious, active learners with individual interests, abilities, and needs. They come to classrooms with varying knowledge, life experiences, and backgrounds. A key component in successfully developing numeracy is making connections to these backgrounds and experiences.

Students construct their understanding of mathematics by developing meaning based on a variety of learning experiences. This meaning is best constructed when learners encounter mathematical experiences that proceed from simple to complex and from the concrete to the abstract. The use of manipulatives, visuals, and a variety of pedagogical approaches can address the diversity of learning styles and developmental stages of students. At all levels of understanding, students benefit from working with a variety of materials, tools, and contexts when constructing meaning about new mathematical ideas. Meaningful student discussions also provide essential links among concrete, pictorial, and symbolic representations of mathematics. The learning environment should value, respect, and address all students’ experiences and ways of thinking so that students are comfortable taking intellectual risks, asking questions, and posing conjectures. Students need to explore mathematics through solving problems in order to continue developing personal strategies and mathematical literacy. It is important to realize that it is acceptable to solve problems in different ways and that solutions may vary depending upon how the problem is understood.

Goals of Mathematics Education

The main goals of mathematics education are to prepare students to
- use mathematics confidently to solve problems
- communicate and reason mathematically
- appreciate and value mathematics
- make connections between mathematics and its applications
- become mathematically literate adults, using mathematics to contribute to society
- commit themselves to lifelong learning

Students who have met these goals will
- gain understanding and appreciation of the contributions of mathematics as a science, a philosophy, and an art
- exhibit a positive attitude toward mathematics
- engage and persevere in mathematical tasks and projects
- contribute to mathematical discussions
- take risks in performing mathematical tasks
- exhibit curiosity about mathematics and situations involving mathematics

In order to assist students in attaining these goals, teachers are encouraged to develop a classroom atmosphere that fosters conceptual understanding. Students should be encouraged to
- take risks
- think and reflect independently
- share and communicate mathematical understanding
- solve problems in individual and group projects
- pursue greater understanding of mathematics
- appreciate the value of mathematics throughout history

Opportunities for Success

A positive attitude has a profound effect on learning. Environments that create a sense of belonging, encourage risk taking, and provide opportunities for student success help develop and maintain positive attitudes and self-confidence. Students with positive attitudes toward learning mathematics are likely to be motivated and prepared to learn, participate willingly in classroom activities, persist in challenging situations, and engage in reflective practices.

To experience success, students must be taught to set achievable goals and assess their progress as they work toward these goals. Striving toward success and becoming autonomous and responsible learners are ongoing, reflective processes that involve revisiting the setting and assessing of personal goals.

Engaging All Learners

*"*No matter how engagement is defined or which dimension is considered, research confirms this truism of education: *The more engaged you are, the more you will learn."* (Hume 2011, 6)

Student engagement is at the core of learning. Engagement in learning occurs when students are provided with opportunities to become more invested in their learning. This is critical for teachers to take into account when planning and implementing instruction. Effective instruction engages, embraces, and supports all learners through a range of learning experiences that are both age and developmentally appropriate.

This curriculum is designed to provide learning opportunities that are equitable, accessible, and inclusive of the many facets of diversity represented in today's classrooms. When teachers know their students as individual learners and as individual people, their students are more likely to be motivated to learn, persist in challenging situations, and apply reflective practices.

Supportive Learning Environments

A supportive and positive learning environment has a profound effect on students' learning. Students need to feel physically, socially, emotionally, and culturally safe in order to take risks with their learning. In classrooms where students feel a sense of belonging, see their teachers' passion for learning and teaching, are encouraged to actively participate, and are challenged appropriately, they are more likely to be successful.

Teachers recognize that not all students progress at the same pace nor are they equally positioned in terms of their prior knowledge of particular concepts, skills, and learning outcomes. Teachers are able to create more equitable access to learning when

- instruction and assessment are flexible and offer multiple means of representation
- students have options to engage in learning through multiple ways
- students can express their knowledge, skills, and understanding in multiple ways

(Hall, Meyer, and Rose 2012)

In a supportive learning environment, teachers plan learning experiences that support *each* student's ability to achieve curriculum outcomes. Teachers use a variety of effective instructional approaches that help students to succeed, such as

- providing a range of learning opportunities that build on individual strengths and prior knowledge
- providing all students with equitable access to appropriate learning strategies, resources, and technology
- involving students in the creation of criteria for assessment and evaluation
- engaging and challenging students through inquiry-based practices
- verbalizing their own thinking to model comprehension strategies and new learning
- balancing individual, small-group, and whole-class learning experiences
- scaffolding instruction and assignments as needed and giving frequent and meaningful descriptive feedback throughout the learning process
- integrating "blended learning" opportunities by including an online environment that extends learning beyond the physical classroom
- encouraging students to take time and to persevere, when appropriate, in order to achieve a particular learning outcome

Multiple Ways of Learning

"Advances in neuroscience and education research over the past 40 years have reshaped our understanding of the learning brain. One of the clearest and most important revelations stemming from brain research is that there is no such thing as a 'regular student.'" (Hall, Meyer, and Rose 2012, 2) Teachers who know their students well are aware of students' individual learning differences and use this understanding to inform instruction and assessment decisions.

The ways in which students make sense of and demonstrate learning vary widely. Individual students tend to have a natural inclination toward one or a few learning styles. Teachers are often able to detect learning strengths and styles through observation and through conversation with students. Teachers can also get a sense of learning styles through an awareness of students' personal interests and talents. Instruction and assessment practices that are designed to account for multiple learning styles create greater opportunities for all students to succeed.

While multiple learning styles are addressed in the classroom, the three most commonly identified are:

- auditory (such as listening to teacher-modelled think-aloud strategies or participating in peer discussion)
- kinesthetic (such as examining artifacts or problem-solving using tools or manipulatives)
- visual (such as reading print and visual texts or viewing video clips)

For additional information, refer to *Frames of Mind: The Theory of Multiple Intelligences* (Gardner 2007) and *How to Differentiate Instruction in Mixed-Ability Classrooms* (Tomlinson 2001).

A Gender-Inclusive Curriculum and Classroom

It is important that the curriculum and classroom climate respect the experiences and values of all students and that learning resources and instructional practices are not gender-biased. Teachers promote gender equity and inclusion in their classrooms when they

- articulate equally high expectations for all students
- provide equal opportunity for input and response from all students
- model gender-fair language, inclusive practices, and respectful listening in their interactions with students
- identify and openly address societal biases with respect to gender and sexual identity

Valuing Diversity: Teaching with Cultural Proficiency

"Instruction that is embedded in socially meaningful contexts, and tasks that are meaningful and relevant to the lives of students, will engage students in high-level problem-solving and reasoning and enhance students' engagement (Frankenstein 1995; Gutstein 2003; Ladson-Billings 1997; Tate 1995)." (Herzig 2005)

Teachers appreciate that students have diverse life and cultural experiences and that individual students bring different prior knowledge to their learning. Teachers can build upon their knowledge of their students as individuals, value their prior experiences, and respond by using a variety of culturally-proficient instruction and assessment practices in order to make learning more engaging, relevant, and accessible for all students. For additional information, refer to *Racial Equity Policy* (Nova Scotia Department of Education 2002) and *Racial Equity / Cultural Proficiency Framework* (Nova Scotia Department of Education 2011).

Students with Language, Communication, and Learning Challenges

Today's classrooms include students who have diverse language backgrounds, abilities, levels of development, and learning challenges. By observing and interacting with students and by conversing with students and/or their families, teachers gain deeper insights into the student as a learner. Teachers can use this awareness to identify and respond to areas where students may need additional support to achieve their learning goals. For students who are experiencing difficulties, it is important that teachers distinguish between those students for whom curriculum content is challenging and those for whom language-based factors are at the root of apparent academic difficulties. Students who are learning English as an additional language may require individual support, particularly in language-based subject areas, while they become more proficient in their English language skills. Teachers understand that many students who appear to be disengaged may be experiencing difficult life or family circumstances, mental health challenges, or low self-esteem, resulting in a loss of confidence that affects their engagement in learning. A caring, supportive teacher demonstrates belief in the students' abilities to learn and uses the students' strengths to create small successes that help nurture engagement in learning and provide a sense of hope.

Students who Demonstrate Exceptional Talents and Giftedness

Modern conceptions of giftedness recognize diversity, multiple forms of giftedness, and inclusivity. Some talents are easily observable in the classroom because they are already well developed and students have opportunities to express them in the curricular and extracurricular activities commonly offered in schools. Other talents only develop if students are exposed to many and various domains and hands-on experiences. Twenty-first century learning supports the thinking that most students are more engaged when learning activities are problem-centred, inquiry-based, and open-ended. Talented and gifted students usually thrive when such learning activities are present. Learning experiences may be enriched by offering a range of activities and resources that require increased cognitive demand and higher-level thinking with different degrees of complexity and abstraction. Teachers can provide further challenges and enhance learning by adjusting the pace of instruction and the breadth and depth of concepts being explored. For additional information, refer to *Gifted Education and Talent Development* (Nova Scotia Department of Education 2010).

Connections across the Curriculum

The teacher should take advantage of the various opportunities available to integrate mathematics and other subjects. This integration not only serves to show students how mathematics is used in daily life, but it helps strengthen the students' understanding of mathematical concepts and provides them with opportunities to practise mathematical skills. There are many possibilities for integrating mathematics in business education, career education, literacy, music, physical education, science, social studies, technology education, and visual arts.

Algebra and Number
30–35 hours

GCO: Students will be expected to develop algebraic reasoning and number sense.

SCO AN01 Students will be expected to demonstrate an understanding of the absolute value of real numbers. [R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

AN01.01 Determine the distance of two real numbers of the form $\pm a$, $a \in R$, from 0 on a number line, and relate this to the absolute value of $a(|a|)$.
AN01.02 Determine the absolute value of a positive or negative real number.
AN01.03 Explain, using examples, how distance between two points on a number line can be expressed in terms of absolute value.
AN01.04 Determine the absolute value of a numerical expression.
AN01.05 Compare and order the absolute values of real numbers in a given set.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
—	**AN01** Students will be expected to demonstrate an understanding of the absolute value of real numbers.	—

Background

The concept of absolute value is new to students in this course. The **absolute value** of a number is its distance from zero on a number line.

Students should notice that the integers 5 and –5, when graphed on a number line, are the same distance from zero. It is important for students to understand that absolute value only asks "how far?" not "in which direction?". Reinforce that distance is always positive. This leads to the definition of the absolute value of any real number a:

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$$

It should be noted that absolute value can also be referred to as the magnitude of the number.

In Mathematics 9, students simplified numerical expressions using the order of operations (9N04). This is now extended to include expressions containing absolute value. When asking students to compare the expressions of 3 – 4(2) and |3 – 4(2)|, they should be reminded to apply the order of operations.

If an expression contains an absolute value expression, students will first simplify the expression within the absolute value bars, find its absolute value, and then simplify the remaining expression.
For example:

$$\begin{aligned} 10-5|1-3(2)| &= 10-5|1-6| \\ &= 10-5|-5| \\ &= 10-5(5) \\ &= 10-25 \\ &= -15 \end{aligned}$$

Students will compare and order absolute values of real numbers. Once students evaluate and compare the absolute values in a given set, they can place the values on a number line to help them order the set. Before ordering the absolute values of real numbers in a given set, it is important to find the absolute value of each of the numbers in the set. Essentially, this means that we are ordering the magnitudes of the numbers and ignoring the signs on the numbers.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

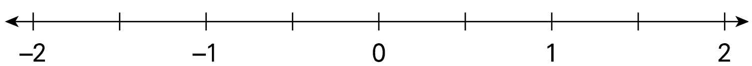
- What integers are two units from 3?
- Discuss what –4 and 4 have in common and how they are different.
- What do you know about the values of A and B if
  (a) $A + B = 0$
  (b) $(A)(B) = 0$
  (c) $\frac{A}{B} = 0$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the value of each of the following:
  (a) $|3-5|$
  (b) $4|3^2-5|$
  (c) $4-|7-5(-2)|$
  (d) $7\left|\frac{2}{5}-5\right|+\left|(-2)^3\right|$

- Place the following numbers on the number line below.
  $A(0.7)$, $B(-1.4)$, $C\left(-\frac{3}{5}\right)$, $D\left(-2\frac{1}{4}\right)$, $E(2)$



  (a) Determine the absolute value of each number.
  (b) Determine the distance between *B* and *E*. How is this distance related to the absolute values of *B* and *E*?
  (c) Determine the distance between *C* and *D*. How is this distance related to the absolute values of *C* and *D*?

- Determine the value of $7|0.4-5|+\left|(-2)^3\right|$

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?

- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students may have difficulty understanding how it is possible for $|x| = -x$. This is because they will automatically think that x is positive. It is a good idea to write a variable on the board and ask students to give you numbers that the variable might represent. Then, when students have included various types of numbers (positive, negative, rational, irrational), ask them to write the absolute value for each of these possibilities. This should lead students to a solid understanding of the definition $|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$.

- Asking students to place positive and negative numbers on a number line and determine the distance each number is from zero is one method of emphasizing that the distance that a number is from zero represents the absolute value, or magnitude, of the number.

- Have students order real numbers in various forms (decimal, fractions, integers, mixed numbers) including absolute values.

- Some possible learning tasks that reinforce working with absolute value are described below.
  - Provide students with a fictitious town having one main street. This street should contain at least eight landmarks located to the right and left of the town square, which is representative of the origin. Each landmark should be given a name (gas station, library, etc.). Ask students to pose three questions where the distance between any two landmarks is requested. They should then answer their student-generated questions.
  - Create a set of cards with a variety of examples involving the absolute values of real numbers and numerical expressions. After arranging students in groups of three, give each student five cards. They will place the cards in a position that produces an ordered set of cards. When all members of the group are finished, have each student explain their reasoning to the other members.
  - Ask students to create a human number line. Each student is given a card containing an absolute value of a real number or numerical expression. They will then order themselves into a line based on the relative size of their number. Ask each student to explain why he or she chose a particular position. As an alternative, a skipping rope or piece of string with clothes pins, paper clips, or binder clips could be used, to which students could attach their numbers in the appropriate position.
  - Divide the class into teams consisting of about five students. Have each team line up. Provide each team with a list of absolute value problems to simplify. The first student in each line should write the answer to one of the absolute value problems using mental mathematics skills. They should then move to the back of the line and the second student should repeat this with a different problem. The first team to complete their questions with the correct answers would be the winners.

- Students will extend the concept of absolute value to include the distance between any two real numbers. This could be first investigated using natural numbers and then extended to integers and real numbers. Completing a table such as the one below should help students recognize that the distance between a and b can be represented by $|a-b|$ or $|b-a|$.

a	***b***	**Distance between *a* and *b***	**Value of $|a-b|$**	**Value of $|b-a|$**
2	6	4	$|2-6|=|-4|=4$	$|6-2|=|4|=4$
–5	–10			
2.68	5.75			

SUGGESTED MODELS AND MANIPULATIVES

- grid paper
- ruler

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- absolute value
- order of operations
- magnitude of a number

Resources/Notes

Internet

- *Sheppard Software* (Sheppard and Chapgar 2014)
  www.sheppardsoftware.com
  This website is an interactive game for ordering the absolute values of real numbers.
  (search: number balls absolute value)

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  – Section 7.1, pp. 358–367

Notes

SCO AN02 Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands. [CN, ME, PS, R, T]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

AN02.01 Compare and order radical expressions with numerical radicands in a given set.
AN02.02 Express an entire radical with a numerical radicand as a mixed radical.
AN02.03 Express a mixed radical with a numerical radicand as an entire radical.
AN02.04 Perform one or more operations to simplify radical expressions with numerical or variable radicands.
AN02.05 Rationalize the denominator of a radical expression with monomial or binomial denominators.
AN02.06 Describe the relationship between rationalizing a binomial denominator of a rational expression and the product of the factors of a difference of squares expression.
AN02.07 Explain, using examples, that $(-x)^2 = x^2$, $\sqrt{x^2} = |x|$, and $\sqrt{x^2} \neq \pm x$.
AN02.08 Identify the values of the variable for which a given radical expression is defined.
AN02.09 Solve a problem that involves radical expressions.

Scope and Sequence

Mathematics 10	Pre-calculus 11	Pre-calculus 12
AN02 Students will be expected to demonstrate an understanding of irrational numbers by representing, identifying, simplifying, and ordering irrational numbers.	**AN02** Students will be expected to solve problems that involve operations on radicals and radical expressions with numerical and variable radicands.	**RF013** Students will be expected to graph and analyze radical functions (limited to functions involving one radical).

Background

In Mathematics 9, students determined the square root of a perfect square and worked with benchmarks to approximate the square root of non-perfect square rational numbers (9N05, 9N06). In Mathematics 10, students worked with mixed and entire radicals, limited to numerical radicands (10AN02). The notation $\sqrt[n]{x}$ was introduced, where the index of the radical was a maximum of 5.

In Mathematics 10, students expressed a radical as either a mixed or entire radical with numerical radicands. Students converted an entire radical such as $\sqrt{98}$ to a mixed radical $\sqrt{98} = \sqrt{7^2 \bullet 2} = 7\sqrt{2}$. Students also converted a mixed radical such as $5\sqrt[3]{2}$ to an entire radical $5\sqrt[3]{2} = \sqrt[3]{2 \bullet 5^3} = \sqrt[3]{250}$.

In this course, students will convert an entire radical with one variable to a mixed radical and will then progress to multiple variables. They will also work backwards and express a mixed radical with a variable as an entire radical.

When simplifying $\sqrt[4]{x^7}$, for example, students can rewrite as $\sqrt[4]{x^7}=\sqrt[4]{x^4 \bullet x^3}=x\sqrt[4]{x^3}$.

In the example $x\sqrt[4]{x^3}$, for the radical to represent a real number, $x \geq 0$ because the index is a real number. When writing $2x\sqrt[3]{5x^2}$ as an entire radical, the variable *x* in the expression $2x\sqrt[3]{5x^2}=\sqrt[3]{2^3 \bullet x^3 \bullet 5 \bullet x^2}=\sqrt[3]{40x^5}$ can be any real number since the index of the radical is an odd number.

It will likely be necessary to review and reinforce the fact that the radicands can be compared, without the use of technology, if they have the same index. It is helpful to rearrange the mixed radical as an entire radical for the purpose of ordering and estimation without the use of technology. When given two right triangles, for example, ask students to determine which hypotenuse length is greater, $3\sqrt{5}$ or $4\sqrt{3}$. Although students could use a calculator to approximate the length, the focus here is to rewrite the numerical radicals in equivalent forms, such as $3\sqrt{5}=\sqrt{3^2 \bullet 5}=\sqrt{45}$ as compared to $4\sqrt{3}=\sqrt{4^2 \bullet 3}=\sqrt{48}$, and to then make a comparison.

In Mathematics 10, students simplified radicals with numerical radicands. They also determined the domain of a variety of functions (RF01). This is their first exposure to simplifying expressions with variable radicands. They will write the restrictions on the variable and then write the expression in its simplest form. For example, when given the expression $\sqrt{2x-6}$, students will determine that since the radicand of a root with an even index must not be negative, we know that $2x-6 \geq 0$ and, therefore, that $2x \geq 6$; $x \geq 3$.

For this outcome, students will practise performing arithmetic operations (adding, subtracting, multiplying, and dividing) on radical expressions, including defining the domain of radicands and learning how to **rationalize a denominator**. This will be extended to binomial as well as monomial expressions that include variables as part of the radicand.

Students should have some exposure to principal and secondary square roots. They should recognize that every positive number has two roots. In other words, if we know that x^2 = 49, then we know that *x* could be either 7 or –7. Thus we could say if x^2 = 49, then $x=\pm\sqrt{49}$. The value $\sqrt{49}=7$ is called the **principal square root** and $-\sqrt{49}=-7$ is the **secondary square root**.

Although students will compare principal and secondary square roots, it is equally important for them to understand why it makes sense to only use the principal square root in certain situations. For example, if students are using the Pythagorean theorem to calculate the length of a leg of a right triangle, length is a positive number, so the square root must also be positive and, therefore, only the principal square root is considered in this situation.

Students are exposed to the concept of absolute value in this course. If this has not been done prior to working with irrational expressions, an introduction to absolute value is important here. Ensure students understand that the absolute value symbol produces a result that is always positive, which is the principal square root. Hence $\sqrt{x^2}=|x|$. This may seem trivial, but it is an important concept for further mathematical study.

For example, if $x = -2$, then $\sqrt{(-2)^2} = \sqrt{4} = 2$; therefore, in this situation $\sqrt{x^2} \neq x$, but it is true that $\sqrt{x^2} = |x|$. Once this idea has been established, students should recognize that in many situations, such as the dimensions of a geometric shape, all variables must represent a positive number. Therefore, although writing $\sqrt{x^2} = x$ is often accepted as correct, it is only accurate when the variable represents a positive number.

Students should be comfortable recognizing that square roots of positive numbers are defined under the set of real numbers. Remind students of this concept by using numerical examples such as $\sqrt{4} = 2$ while $\sqrt{-4}$ is not defined under the set of real numbers. Students should recognize that if a radical has an even index, the radicand must be non-negative. If a radical has an odd index, the radicand can be any real number, including negative numbers.

Students should understand what happens if the radicand is variable in nature, such as $\sqrt{x+5}$; that the domain of a square root function is limited to values for which the function has meaning. When variables are part of the radicand, they will be defined to ensure this is true. For example, for $\sqrt{x-2}$, the radicand, $x - 2$, must be greater than zero, so x will be defined as $x \geq 2$. It is important to use examples and to allow students to intuitively investigate variable expressions as radicands before progressing to actually solving an inequality algebraically. While students solved inequalities in Mathematics 9 (PR04), it may be necessary to review the various rules used to solve inequalities, such as the fact that when multiplying or dividing by a negative, the inequality reverses. For example, $\sqrt{50-10x}$ would be defined only when $50 - 10x \geq 0$ or $-10x \geq -50$, and therefore $x \leq 5$.

It is important for students to recognize that even when adding or subtracting like radicals, the solution can require further simplifying. When subtracting the expression $3\sqrt{8} - 7\sqrt{8}$, for example, the solution $-4\sqrt{8}$ can be simplified to $-8\sqrt{2}$. It would be good practice to simplify the radical before like terms are combined. This is especially beneficial when working with large numerical radicands. Students should also be exposed to examples such as $2\sqrt{18} + 3\sqrt{50} - 5\sqrt{2}$ or $4\sqrt[3]{54} + \sqrt[3]{-45} - 3\sqrt[3]{2}$ in which it is necessary to simplify one or more radicals in order to complete the addition and/or subtraction operations. Students will add and subtract radicals that contain numerical and variable radicands before they move into multiplication and division.

In addition to adding and subtracting radicals, students will multiply and divide radicals beginning with numerical radicands. To demonstrate the multiplication property of radicals, reiterate the relationship between a radical and a power with rational exponents.

$$3^{\frac{1}{2}} \cdot 5^{\frac{1}{2}} = (3 \cdot 5)^{\frac{1}{2}} = 15^{\frac{1}{2}} \Leftrightarrow \sqrt{3} \cdot \sqrt{5} = \sqrt{3 \cdot 5} = \sqrt{15}$$

In Mathematics 10, students applied the laws of exponents to rational exponents (AN03). Encourage students to look for a pattern through the use of several examples. The multiplicative property of radicals can then be introduced to students, $\sqrt{a} \bullet \sqrt{b} = \sqrt{ab}$ where $a \geq 0$ and $b \geq 0$. This property can also be used to discuss why two radicals with the same index can be multiplied. Although students were exposed to this property in Mathematics 10, it was used exclusively for the purpose of expressing a radical as a mixed or entire radical (AN02).

Students should be given an opportunity to further explore the product rule for radicals (i.e., $c\sqrt{a} \bullet d\sqrt{b} = cd\sqrt{ab}$) and the commutative property of multiplication.

Students will also be exposed to examples in which the index is not 2. They should be introduced to the rule $c\sqrt[n]{a} \bullet d\sqrt[n]{b} = cd\sqrt[n]{ab}$, where *n* is a natural number and *c*, *d*, *a*, and *b* are real numbers. They should recognize that if *n* is even then $a \geq 0$ and $b \geq 0$.

Students will explore how multiplying and dividing algebraic expressions with radicals is similar to multiplying and dividing numerical expressions with radical values.

If a radical appears in the denominator, the expression is most often rationalized to produce a rational denominator. When students study calculus they will sometimes need to rationalize the numerator but for purposes of this course students will rationalize the denominator. Rationalizing the denominator is done by multiplying both the numerator and denominator by the radical in the denominator. In effect, the expression is multiplied by 1, and therefore the value of the expression is unchanged. For example, the expression below, which has a monomial denominator, is rationalized as follows:

$$\frac{2}{\sqrt{3}} = \left(\frac{2}{\sqrt{3}}\right) \bullet \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{2\sqrt{3}}{3} \text{ or } \frac{2}{3}\sqrt{3}$$

Examples such as $\frac{2\sqrt{3}}{7\sqrt{5}}$, in which the denominator is a mixed radical should be included. Encourage students to think about what they would have to multiply by to rationalize the denominator. The initial tendency may be to multiply the numerator and denominator by $7\sqrt{5}$. Although it is not the most efficient strategy, this impulse would still be correct. However, the resulting expression will have to be simplified.

They should understand multiplying by $\frac{\sqrt{5}}{\sqrt{5}}$ will rationalize the denominator.

Students should first be exposed to expressions with entire radicals in both the numerator and the denominator. This should then be extended to include mixed radicals and examples in which there is more than one term in the numerator. It is the students' choice whether they simplify before or after they rationalize the denominator. When working with larger numbers, however, simplifying first would allow them to work with smaller numbers.

If the denominator is a binomial, the **conjugate** of the binomial must be used to rationalize the denominator. The conjugate of a binomial has the sign between the terms changed from positive to negative or from negative to positive. For example, the conjugate of (2*a* – 3) is (2*a* + 3), and when these two binomials are multiplied, the resulting quadratic is the difference of squares. Students have seen the conjugate in Mathematics 10 in relation to the difference of squares. To rationalize a binomial denominator, the numerator and denominator are both multiplied by the conjugate of the denominator.

$$\frac{3}{\sqrt{5}-1}$$

$$=\left(\frac{3}{\sqrt{5}-1}\right)\bullet\left(\frac{\sqrt{5}+1}{\sqrt{5}+1}\right)$$

$$=\frac{3\sqrt{5}+3}{5+\sqrt{5}-\sqrt{5}-1}$$

$$=\frac{3\sqrt{5}+3}{4} \text{ or } \frac{3}{4}\sqrt{5}+\frac{3}{4}$$

When asked to rationalize $\frac{3}{\sqrt{5}-1}$, both $\frac{3\sqrt{5}+3}{4}$ and $\frac{3}{4}\sqrt{5}+\frac{3}{4}$ are considered to be equally correct. Students should feel comfortable with both methods of writing the final answer as there will be situations where either would be advantageous.

It would be expected that students would simplify fractions where possible. For example, if rationalizing an expression resulted in the answer $\frac{8\sqrt{5}-6}{4}$, students would be expected to write either $\frac{4\sqrt{5}-3}{2}$ or $\frac{4\sqrt{5}}{2}-\frac{3}{2}$, which could also be written correctly as $2\sqrt{5}-\frac{3}{2}$.

Students should also be exposed to expressions in which both terms in the binomial are irrational such as $\frac{2}{\sqrt{5}-3\sqrt{2}}$. Expressions involving variable numerators and denominators should be explored in a similar fashion.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Multiply the following: $(2x + 1)(2x - 1)$.
- Write $\sqrt{50}$ as a mixed radical.
- Write $2\sqrt{3}$ as an entire radical.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Order from least to greatest.

  $\sqrt{5}$, $2\sqrt{3}$, $-\sqrt{3}$, $\sqrt{2^2}$, $\dfrac{\sqrt{9}}{3}$

- Write each of the following in the alternate form. If entire, write as mixed; if mixed, write as entire.
  - (a) $2\sqrt{3}$
  - (b) $\sqrt{50}$
  - (c) $\sqrt{90x^3}$
  - (d) $5m^3\sqrt{2}$
  - (e) $3x\sqrt{7x}$

- Simplify the following:
  - (a) $\sqrt{3}\,(2\sqrt{6}-4\sqrt{5})$
  - (b) $\dfrac{-12\sqrt{22}}{4\sqrt{11}}$
  - (c) $\sqrt{x}(3\sqrt{x}-4\sqrt{y})$
  - (d) $(7\sqrt{x}+3)(2\sqrt{x}-6)$
  - (e) $\dfrac{12x\sqrt{12y}}{3\sqrt{3y}}$

- Match the following equivalent expressions.

(a) $(-6)^2$	(i) $\sqrt{12}$
(b) $\lvert 4 \rvert$	(ii) $\sqrt{16}$
(c) $2\sqrt{3}$	(iii) $\dfrac{2}{\sqrt{5}}$
(d) $\sqrt{15}$	(iv) 6^2
(e) $\dfrac{2\sqrt{5}}{5}$	(v) $\sqrt{3} \bullet \sqrt{5}$

- Define values of the variable for which the radicand is non-negative.

  (a) $\sqrt{2x+1}$ (b) $\dfrac{3}{\sqrt{x-9}}$ (c) $\dfrac{4-\sqrt{6x}}{8}$

- Sheldon told Leonard that $\sqrt{x^2} = x$. Use specific examples to show whether you agree or disagree with Sheldon's statement.
- Evaluate each of the following, if possible: $\sqrt{16}$, $\sqrt[4]{-81}$, $\sqrt[3]{-27}$, $\sqrt[3]{8}$, $\sqrt[6]{-64}$, $\sqrt[5]{-32}$.
- Explain why $\sqrt{-16}$ is undefined, while $\sqrt[3]{-64}$ is defined under the set of real numbers.
- Explain why $(-9)^{\frac{1}{2}}$ is undefined, while $(-125)^{\frac{1}{3}}$ is defined under the set of real numbers.
- Complete the following charts and make predictions about the restrictions on each variable.

Chart 1

x	$\sqrt{x}$	$\frac{1}{x}$	$\frac{1}{\sqrt{x}}$
–3			
–2			
–1			
0			
1			
2			
3			

(a) What values of *x* were undefined?
(b) What values of *x* were defined?
(c) Is the restriction different if the radical expression is in the denominator?

Chart 2

x	$\sqrt{x+1}$	$\frac{1}{x+1}$	$\frac{1}{\sqrt{x+1}}$
–3			
–2			
–1			
0			
1			
2			
3			

(a) What values of *x* were undefined?
(b) What values of *x* were defined?
(c) Is the restriction different if the radical expression is in the denominator?

Chart 3

x	$\sqrt{x-1}$	$\frac{1}{x-1}$	$\frac{1}{\sqrt{x-1}}$
–3			
–2			
–1			
0			
1			
2			
3			

(a) What values of *x* were undefined?
(b) What values of *x* were defined?
(c) Is the restriction different if the radical expression is in the denominator?

- For each of the following examples, answer the questions listed below.

  (a) $\sqrt{2x-6}$ (b) $2\sqrt{x+1}$ (c) $\sqrt[3]{2x+5}$

  (i) What values of *x* were undefined? What values of *x* were defined?
  (ii) Is the restriction different if the radical expression is in the denominator?
  (iii) How could solving inequalities help when determining the restriction?

- The voltage (*V*) required for a circuit is given by $V=\sqrt{PR}$, where *P* is the power in watts and *R* is the resistance in ohms. How many more volts are needed to light a 100-W bulb than a 75-W bulb if the resistance for both is 100 ohms? Solve the problem in exact and approximate form.

- Explore how adding and subtracting algebraic expressions with radicals is similar to adding and subtracting numerical expressions with radical values. Use examples similar to $\sqrt{8x^3}-4\sqrt{2x}$, $-2\sqrt[3]{16x^4}+5x\sqrt[3]{54x}$, and $2\sqrt{45}-6\sqrt{20}$ to explain your reasoning.

- Identify and correct errors in the simplifications shown below:

  (a) $25\sqrt{5}+13\sqrt{5}$
  $=38\sqrt{10}$

  (b) $\sqrt{18x^3}+2\sqrt{8x^3}$
  $=3\sqrt{2x^3}+4\sqrt{2x^3}$
  $=7\sqrt{4x^6}$
  $=14x^3$

- Simplify each of the following:
  (a) $2\sqrt{18}+9\sqrt{7}-\sqrt{63}$
  (b) $6\sqrt{32x^3}-5\sqrt{8x^3}+3\sqrt{2x^3}$
  (c) $-5\sqrt[3]{256x}+\sqrt[3]{192x^4}$

- Explore how multiplying and dividing algebraic expressions with radicals is similar to multiplying and dividing numerical expressions with radical values. Use examples similar to $\left(-4\sqrt{12}\right)\bullet\left(-2\sqrt{18}\right)$ and $\left(-4\sqrt{x}\right)\bullet\left(-2\sqrt{x^2}\right)$ where $x\geq 0$, and $\left(-4\sqrt[3]{x}\right)\bullet\left(-2\sqrt[3]{x^2}\right)$ to explain your reasoning.

- When is it necessary to use the distributive property to multiply expressions that contain radicals? Create an example and show the solution.

- Simplify the following:
  (a) $(\sqrt{2}+\sqrt{6})^2$
  (b) $(3\sqrt{8}-4)(2+7\sqrt{3})$
  (c) $(\sqrt{20}+\sqrt{24})(3\sqrt{12}-5\sqrt{32})$
  (d) $-2\sqrt[3]{12}(4\sqrt[3]{2}-5\sqrt[3]{9})$
  (e) $(-3\sqrt{x})(6\sqrt{x^3})$
  (f) $(3\sqrt{x}+2)(3-5\sqrt{x})$

- Rationalize the denominator for each of the following expressions:

  (a) $\dfrac{\sqrt{7}}{1+\sqrt{7}}$ (c) $\dfrac{b}{a+\sqrt{b}}$

  (b) $\dfrac{2+3\sqrt{5}}{2\sqrt{5}-4}$ (d) $\dfrac{x+3\sqrt{y}}{\sqrt{y}-x}$

- When asked to rationalize the denominator in the expression $\dfrac{4}{2+\sqrt{7}}$, Amal said he could just multiply the expression by $\dfrac{\sqrt{7}}{\sqrt{7}}$. Why is he not correct?

- Tara determines the restrictions on the values for *x* in the radical expression $\sqrt{3-7x}$ as follows:

  $$3 - 7x > 0$$
  $$-7x > -3$$
  $$x > \frac{3}{7}$$

  (a) Identify, explain, and correct any errors.
  (b) Explain why radical expressions with variables in the radicand have restrictions.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- When identifying the possible values of a variable for a given radical expression, have students use graphing technology to graph the expressions and investigate where the corresponding function is defined.

- When asked to simplify $\sqrt{x^2}$, students may initially conclude that $\sqrt{x^2} = x$. Prompt student discussion using the following questions:
  - Does this happen with all positive values of *x*?
  - Does this happen with all negative values of *x*?

- Ask students to simplify $\sqrt{(-5)^2}$. The following is a sample of student answers:

  $\sqrt{(-5)^2} = (-5)^{\frac{2}{2}} = (-5)^1 = -5$ $\qquad$ $\sqrt{(-5)^2} = \sqrt{25} = 5$

  This is an opportunity for discussion around the correct answer when the value of *x* is negative. Some students may challenge the incorrect solution provided in column one above. **It is important for students to recognize that when *a* is a negative number, $a^{\frac{m}{n}}$ is not defined because it is not possible to define such expressions consistently.** Students can compare this value when *n* is even to when it is odd (i.e., $(-4)^{\frac{1}{2}}$ is undefined under the set of real numbers, but $(-8)^{\frac{1}{3}}$ is defined).

- This is a great opportunity for discussion of the correct answer when the value of *x* is negative. Although it is common for students to replace $\sqrt{(x)^2}$ with *x*, students must recognize that this is correct only when $x \geq 0$. They should also understand that $\sqrt{(x)^2}$ is equivalent to –*x* when $x \leq 0$. Consider the following:

  When *x* = 4:

  $$\sqrt{x^2} = \sqrt{(4)^2}$$
  $$\sqrt{x^2} = \sqrt{16}$$
  $$\sqrt{x^2} = 4$$
  $$\therefore \sqrt{x^2} = x$$

  When *x* = –4:

  $$\sqrt{x^2} = \sqrt{(-4)^2}$$
  $$\sqrt{x^2} = \sqrt{16}$$
  $$\sqrt{x^2} = 4$$
  $$\therefore \sqrt{x^2} = -x$$
  $$\therefore \sqrt{x^2} \neq x$$

- Introduce the addition and subtraction of radicals that contain numerical and variable radicands by asking students to add $2x + 3x$. Use leading questions such as the following:
  - What process is involved in the addition of monomials?
  - How is this expression similar to $2\sqrt{7}+3\sqrt{7}$?
  - What are like radicals?
  - How can this process be applied to radicals?

  The goal is for students to understand that adding and subtracting radical expressions is comparable to combining variable expressions with like terms. It is necessary for the radical to have the same index and the same radicand. Students can then apply the same strategies with indices other than 2.

- The distributive property can also be applied when simplifying sums and differences of radical expressions. Ask students how to rewrite $2x + 3x$ in another form, namely $(2 + 3)x$. Similarly, $2\sqrt{7}+3\sqrt{7}$ can be expressed as $(2+3)\sqrt{7}$ or $5\sqrt{7}$. Check students' understanding by asking them to express $-4\sqrt{3}$ as the sum of two like radicals.

- Common errors occur when adding or subtracting radicals. When asked to add $4+2\sqrt{3}$ for example, students may write $6\sqrt{3}$. Students should be encouraged to check their answers by expressing the sum/difference using the distributive property. Another common error occurs when students incorrectly apply the operations of addition and subtraction to the radicands. The value of $31\sqrt{3}+17\sqrt{3}$ for example, does not equal $48\sqrt{6}$. This error may occur more often once students have been introduced to the multiplication of radicals.

- To explore the product rule for radicals (i.e., $c\sqrt{a} \bullet d\sqrt{b} = cd\sqrt{ab}$) and the commutative property of multiplication, ask students to rewrite the expression $2(3^{\frac{1}{2}}) \bullet 5(6^{\frac{1}{2}})$ using radicals to generalize a pattern. The approaches students take may vary but the result should be the same. Consider the following example:

$2(3^{\frac{1}{2}}) \times 5(6^{\frac{1}{2}})$	$2(\sqrt{3}) \times 5(\sqrt{6})$
$(2\times5)(3^{\frac{1}{2}}\times6^{\frac{1}{2}})$	$(2\times5)(\sqrt{3}\times\sqrt{6})$
$(10)(3\times6)^{\frac{1}{2}}$	$(10)(\sqrt{3\times6})$
$10(18)^{\frac{1}{2}}$	$10\sqrt{18}$

- Advise students that they are less likely to make simplification errors if they simplify radicals before multiplying. An example such as the following could be used to illustrate the two methods.

  Multiply first: $\sqrt{80} \bullet \sqrt{12} = \sqrt{960} = 8\sqrt{15}$
  Simplify first: $\sqrt{80} \bullet \sqrt{12} = 4\sqrt{5} \bullet 2\sqrt{3} = 8\sqrt{15}$

  Ask students which method they prefer and why.

  Students will also multiply radicals using the distributive property. It may be helpful to walk them through the similarities between multiplying radical expressions and multiplying polynomials.

- Caution students to write the root sign so that it is clear what the radicand is. For example be careful not to write $(2x+1)^{\frac{1}{2}}$ as $\sqrt{2x}+1$.

- This would be a good opportunity to reinforce the commutative and associative property of multiplication so that students can avoid common errors when moving forward. When multiplying $\sqrt{5} \bullet 3$, for example, students may write $\sqrt{15}$. They may understand their error if they apply the commutative property to rewrite the expression as $3\sqrt{5}$. Another error occurs when students are asked to multiply an expression such as $3\sqrt{5} \bullet \sqrt{6}$ and their result is $\sqrt{90}$. This can be avoided if the associative property is used to re-order the expression as $3(\sqrt{5} \bullet \sqrt{6}) = 3\sqrt{30}$.

- The rules of exponents should be integrated when introducing the division of radicals. For example:

$$\left(\frac{4}{9}\right)^{\frac{1}{2}} = \frac{4^{\frac{1}{2}}}{9^{\frac{1}{2}}} \Leftrightarrow \sqrt{\frac{4}{9}} = \frac{\sqrt{4}}{\sqrt{9}} \qquad \left(\frac{8}{27}\right)^{\frac{1}{3}} = \frac{8^{\frac{1}{3}}}{27^{\frac{1}{3}}} \Leftrightarrow \sqrt[3]{\frac{8}{27}} = \frac{\sqrt[3]{8}}{\sqrt[3]{27}}$$

- Ask students to predict the rule when dividing radicals. Using the examples, reinforce the understanding for students that they can only divide radicals that have the same index. They should recognize that the quotient rule of radicals states that the *n*th root of a quotient is the quotient of the *n*th root. In other words, $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \sqrt[n]{\frac{a}{b}}$ where n is a natural number, a and b are real numbers, and $b \neq 0$. Remind students if n is even, then $a \geq 0$ and $b > 0$.

- Students should be exposed to a variety of cases when simplifying radicals with fractions. Include examples such as the following:

$$\sqrt{\frac{25}{4}} = \frac{\sqrt{25}}{\sqrt{4}} = \frac{5}{2} \qquad \frac{\sqrt{12}}{\sqrt{6}} = \sqrt{\frac{12}{6}} = \sqrt{2} \qquad \sqrt{\frac{12}{5}} = \frac{\sqrt{12}}{\sqrt{5}} = \frac{2\sqrt{3}}{\sqrt{5}}$$

  As students simplify radicals such as those above, they should ask themselves the following questions:
  - Is the denominator a perfect root?
  - Can the numerator and denominator divide into a rational number?
  - Will the denominator have a radical when simplified?

  Conclude this task by using an example such as $\frac{2\sqrt{3}}{\sqrt{5}}$, and asking students what they can multiply the numerator and denominator by that results in a rational denominator.

- When students encounter a rationalization problem where the index is greater than 2, such as $\frac{\sqrt[3]{2}}{\sqrt[3]{5}}$, they may initially multiply the numerator and denominator by the term $\sqrt[3]{5}$. They will recognize a radical still remains in the denominator. It is important that you ask them to do this even if none of the students make that assumption and to then look at the result, because this will reinforce the understanding of rationalizing the denominator. Ask for other suggestions. Students will explore the various possibilities and determine that when rationalizing the denominator, the root should guide student choices. They will discover that they need to multiply numerator and denominator by a value where the powers of the radicand in the denominator will equal the index of the root. Therefore, $\frac{\sqrt[3]{2}}{\sqrt[3]{5}} \bullet \left(\frac{\sqrt[3]{5^2}}{\sqrt[3]{5^2}}\right)$ results in a rational denominator.

This strategy can be applied to variable monomial denominators. By giving students an example like $\frac{\sqrt[4]{7x}}{\sqrt[4]{27x^2y^1}}$, the index will help students determine which expression they should use to rationalize the denominator. In other words, $\frac{\sqrt[4]{7x}}{\sqrt[4]{3^3x^2y^1}} \bullet \frac{\sqrt[4]{3^1x^2y^3}}{\sqrt[4]{3^1x^2y^3}}$ will produce a rational denominator.

- Students should try to simplify an expression, such as $\frac{1}{2+\sqrt{3}}$, in which it is necessary to rationalize a binomial denominator. Initially, they may think they can multiply the numerator and denominator by $\sqrt{3}$, as in their previous work with monomial denominators. As they explore this route, however, students should discover that $\frac{1}{2+\sqrt{3}} \bullet \left(\frac{\sqrt{3}}{\sqrt{3}}\right) = \frac{\sqrt{3}}{2\sqrt{3}+3}$ and the denominator still contains a radical. Prompt them to multiply the numerator and denominator by the conjugate of $2+\sqrt{3}$. This is similar to multiplying the factors of a difference of squares expression.

- Some possible tasks that reinforce working with radical expressions are described below.

  - Provide each student with radical expressions having numerical radicands written on a card and have them arrange themselves in a line in order of increasing value.

  - Provide students with blank cards and a list of entire radicals. On one card have students write the entire radical, and on another have them write the corresponding mixed radical. Use these cards in matching games such as Memory, Go Fish, and Find Your Match.

  - Set up a clothesline in the classroom to represent a number line with several benchmarks identified. Give each student a card with an expression of mixed or entire radical. Ask them to pin the card along the number line. They should be able to explain why they placed the card in that position.

  - Divide students into groups of 3–4 students. Each group will be given a deck of cards. Each card will have a different mixed radical. The students in each group will then work together to sort the cards from largest to smallest. The first group with the cards sorted in the correct order wins the competition.

  - Students can play the Radical Matching Game in groups of two. Give students a deck of cards containing pairs that display equivalent mixed radicals and entire radicals. All cards should be placed face down on the table. The first student turns over two of the cards looking for a pair. If a pair turns up, the student removes the cards and goes again. If the overturned cards do not form a pair, it is the other player's turn. The player with the most matches at the end of the game wins.

- For the learning task Sticky Bars, present students with a selected response question for which they could be expected to convert an entire radical with a variable radicand to a mixed radical, or vice versa. The answer is anonymously recorded on a sticky note and submitted to the teacher. The teacher or student volunteer arranges the sticky notes on a wall as a bar graph representing the different student responses. Have a discussion regarding why students may have selected the answers they did.

- In groups of two, ask students to participate in the task Pass the Problem. Give each pair of students a simplification problem that involves rationalizing the denominator. Ask one student to write the first line of the solution and pass it to the second student. The second student will verify the work and check for errors. If there is an error present, ask students to discuss the error and why it occurred. The second student will then write the next line of the solution and pass it to their partner. This process continues until the solution is complete.

Suggested Models and Manipulatives

- grid paper
- scissors

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- conjugate
- principle square root
- rationalize
- secondary square root

Resources/Notes

Internet

- *Professional Learning K–12*, "Radicals" (Government of Newfoundland and Labrador 2013)
  http://www.k12pl.nl.ca/curr/10-12/math/math2201/classroomclips/rad.html
  The Radicals clip demonstrates students performing operations to simplify a radical expression using the task Commit and Toss.

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 5.1 and 5.2, pp. 272–293

Notes

SCO AN03 Students will be expected to solve problems that involve radical equations (limited to square roots). [C, PS, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(*It is intended that the equations will have no more than two radicals.*)

AN03.01 Determine any restrictions on values for the variable in a radical equation.

AN03.02 Determine the roots of a radical equation algebraically, and explain the process used to solve the equation.

AN03.03 Verify, by substitution, that the values determined in solving a radical equation algebraically are roots of the equation.

AN03.04 Explain why some roots determined in solving a radical equation algebraically are extraneous.

AN03.05 Solve problems by modelling a situation using a radical equation.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
AN02 Students will be expected to demonstrate an understanding of irrational numbers by representing, identifying, simplifying, and ordering irrational numbers. (M10)* **AN03** Students will be expected to demonstrate an understanding of powers with integral and rational components. (M10) **AN04** Students will be expected to demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials and trinomials), concretely, pictorially, and symbolically. (M10)	**AN03** Students will be expected to solve problems that involve radical equations (limited to square roots).	**RF13** Students will be expected to graph and analyze radical functions (limited to functions involving one radical).

Mathematics 10 / Mathematics 11 (continued)	**Pre-calculus 11**	**Pre-calculus 12**
RF02 Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**		

* M10—Mathematics 10
** M11—Mathematics 11

Background

In previous grades, students became familiar with perfect squares and square roots. This outcome extends this learning to equations in which the radical must first be isolated before it can be solved.

In the previous outcome (AN02), students were introduced to the concept that, when the index was even, the radicand must be positive. When solving for a variable in an equation, the solution(s) must meet the restriction(s) of the original equation.

In Mathematics 11, students learned the similarities and differences between "zeros of a function, " "roots of an equation, " and "x-intercepts" of a graph (11RF02). In this course, students will learn that there are conditions in which one or more roots might not satisfy the original equation, and these are called **extraneous roots**.

These extraneous roots can occur when both sides are squared and then solved using factoring techniques, the quadratic formula, or by taking the square root of the number.

When solving for a variable, the roots must meet the restriction and be verified by substituting back into the original equation. For example,

Solve for *x*.

$x+2\sqrt{(x-1)}=9$ (restriction $x\geq 1$)

$2\sqrt{(x-1)}=9-x$

$(2\sqrt{(x-1)})^2=(9-x)^2$

$4x-4=81-18x+x^2$

$0=(x-5)(x-17)$

$0=\{5,\ 17\}$

Both 5 *and* $17\geq 1$ so both roots satisfy the restriction.

Verify *x* = 5.

$x+2\sqrt{x-1}=9$

left side

$5+2\sqrt{5-1}$

$=5+2(2)$

$=9$

right side

$=9$

Left side = right side
$\therefore$ 5 is a valid solution.

Verify x = 17.

$x+2\sqrt{x-1}=9$

left side

$17+2\sqrt{17-1}$

$=17+2(4)$

$=25$

right side

$=25$

Left side $\neq$ right side
$\therefore$ 17 is an extraneous root.

Extraneous roots occur because squaring both sides and solving the equation may result in roots that do not satisfy the original equation. As students solve equations, reinforce the importance of checking that the value is a solution to the original equation. When appropriate, students should state the root is extraneous. Any extraneous roots are rejected as answers.

Extraneous roots should not be confused with answers that do not make sense within the context of a word problem. For example,

- The depth of a particular submarine can be expressed as a function of time. The formula is $d(t) = t^2 + 9d - 36$. When will the submarine surface next? Solving for t with $d(t) = 0$ (depth is 0 at surface): $0 = t^2 + 9t - 36$; $0 = (t - 3)(t + 12)$; $t = 3$, $t = -12$.

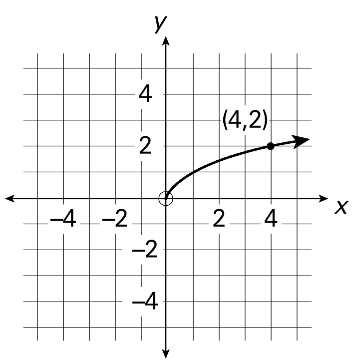
Even though verification shows –12 to be valid, a time of negative 12 seconds does not make sense, so this answer is incorrect and therefore considered to be an **inadmissible root**.

Students will solve radical equations involving square roots. For the scope of this course, it is intended that the equations will have no more than two radicals and the radicand will contain variables that are first- or second-degree polynomials. Students will be responsible for solving equations resulting in a linear or quadratic equation.

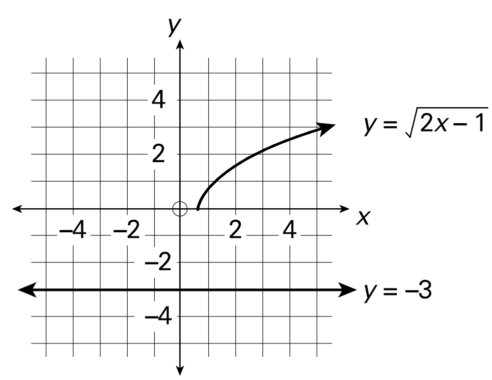
Earlier in this course (AN02), students determined restrictions within the real number system on a variable in a radical that had an even index. They will continue to determine the restrictions before solving a radical equation.

Using technology students will compare a radical equation to its graph to develop an understanding of restrictions for the variable and of the points that satisfy the equation.

For example the graph of $y = \sqrt{x}$ below would illustrate that the answer to the equation $\sqrt{x} = 2$ is 4 and that $x \geq 0$.



Focusing on the point (4, 2), students will be asked how they would algebraically solve the equation given only the *y*-coordinate 2. When solving $2 = \sqrt{x}$, the value of *x* can be determined by inspection. Students could also use the idea that squaring a number is the inverse operation of taking the square root. This technique may seem straightforward, but students should be exposed to equations where the value is not a solution to the original equation. Consider the example $\sqrt{2x-1} = -3$. The left-hand side of the equation calls for a positive square root, but the right-hand side of the equation is negative. Intuitively, there can be no solution. This can also be illustrated graphically, as shown below.



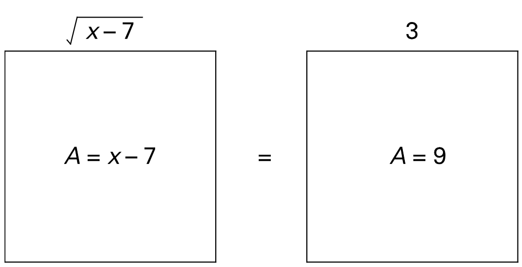
Students should recognize, that given the y-coordinate of –3, there is no possible x-coordinate that satisfies the equation. However, squaring both sides of the equation results in $(\sqrt{2x-1})^2 = (-3)^2$; $2x - 1 = 9$; $2x = 10$; or $x = 5$. This cannot be correct, as both substitution and the graph have shown that the equation has no solution. This is a great lead-in to the concept of extraneous roots.

Another strategy students can use to solve a radical equation involves applying a power that will eliminate the radical expression on both sides of the equation. To solve $\sqrt{x+1} = 4$ for example, students would first rewrite the radical expression with a rational exponent, resulting in $(x+1)^{\frac{1}{2}} = 4$. Squaring both sides of this equation yields $\left((x+1)^{\frac{1}{2}}\right)^2 = 4^2$ and leads to an equation without radicals, $x + 1 = 16$ or $x = 15$. Or they could rewrite the equation $\sqrt{x+1} = 4$ so that the exponents were the same $(x+1)^{\frac{1}{2}} = 16^{\frac{1}{2}}$, which would leads to an equation without radicals, $x + 1 = 16$ or $x = 15$.

Expose students to equations where the radical is not isolated. In such situations, make a comparison to solving a linear equation. Students should recognize that solving an equation such as $3 + \sqrt{2x+1} = 7$ follows a process that is similar to solving $3 + K = 7$.

The area model can also be used to solve radical equations. In Mathematics 8, students viewed the area of the square as a perfect square number, and the side length of the square as its square root (N01). Recall that, if a square has an area of 9, then its side has a length of 3. Similarly, if a square has an area of 3, then its side has a length of $\sqrt{3}$. Teachers should prompt discussion about the side length of a square if its area is x. Consider the following example: Solve $\sqrt{x-7} = 3$.

Using the area model, students should use two congruent squares and label the dimensions of one square as $\sqrt{x-7}$ and the dimension of the other square as 3. They can then determine the area of each square.



Students should recognize that the goal is to determine the value of x which results in the same area for both congruent squares. Solving the equation $x - 7 = 9$ results in $x = 16$. This representation helps students visualize what each equation is describing. Encourage students to check their answers by substituting the value back into the original equation.

Students will also solve radical equations that result in quadratic equations.

Expose students to questions that have no solutions, one solution, or two solutions.

Students should proceed to solve the radical equation algebraically by squaring both sides of the equation. They can solve the resulting quadratic equation using the method of their choice (i.e., graphing, factoring, completing the square—if done prior to this outcome—or using the quadratic formula).

For example, present an equation such as $\sqrt{2x+1}=x-1$. Squaring both sides of this equation yields

$$\left(\sqrt{2x+1}\right)^2=(x-1)^2$$
$$2x+1=x^2-2x+1$$
$$0=x^2-4x$$
$$0=x(x-4)$$
$$x=0;\ x=4$$

Check *x* = 0.

$$\sqrt{2x+1}=x-1$$
$$\sqrt{2(0)+1}=0-1$$
$$\sqrt{1}\neq -1$$

$\therefore x=0$ is extraneous

Check *x* = 4.

$$\sqrt{2x+1}=x-1$$
$$\sqrt{2(4)+1}=4-1$$
$$\sqrt{9}=3$$

$\therefore x=4$ is the only solution to the equation.

Have students solve equations where the radicand is a quadratic. For example,

$$4-\sqrt{x^2-3x}=x+1$$
$$3-x=\sqrt{x^2-3x}$$
$$(3-x)^2=\left(\sqrt{x^2-3x}\right)^2$$
$$9-6x+x^2=x^2-3x$$
$$9-6x=-3x$$
$$9=3x$$
$$3=x$$

Check *x* = 3.

$$4-\sqrt{x^2-3x}=x+1$$
$$4-\sqrt{3^2-3(3)}=3+1$$
$$4-\sqrt{0}=4$$
$$4=4$$

$\therefore x=3$ is the only solution to the equation.

Students will also solve equations that involve two radical expressions. It is recommended in this situation that students isolate the most complicated radical before squaring. As students square both sides in an equation such as $\sqrt{x+7}=2\sqrt{x}+1$, they should recognize the resulting equation still contains a radical. Therefore, students will need to repeat the process of isolating the radical term and squaring both sides of the equation again.

$$\sqrt{x+7}=2\sqrt{x}+1$$
$$(\sqrt{x+7})^2=(2\sqrt{x}+1)^2$$
$$x+7=4x+4\sqrt{x}+1$$
$$-3x+6=4\sqrt{x}$$
$$(-3x+6)^2=(4\sqrt{x})^2$$
$$9x^2-36x+36=16x$$
$$9x^2-52x+36=0$$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$x=\frac{52\pm\sqrt{(-52)^2-4(9)(36)}}{2(9)}$$
$$x=\frac{52\pm\sqrt{1408}}{2(9)}=\frac{52\pm8\sqrt{22}}{18}=\frac{26\pm4\sqrt{22}}{9}$$
$$x=\frac{52\pm37.5233}{18}=4.9735 \text{ and } 0.8043$$

Check x = 4.9735.

$$\sqrt{(4.9735)+7}\neq2\sqrt{(4.9735)}+1$$
$$\sqrt{11.9735}\neq2\sqrt{4.9735}+1$$
$$3.4603\neq4.4603+1$$

$\therefore x=\dfrac{26+4\sqrt{22}}{9}$ is extraneous

Check x = 0.8043.

$$\sqrt{(0.8043)+7}=2\sqrt{(0.8043)}+1$$
$$\sqrt{7.8043}=2\sqrt{0.8043}+1$$
$$2.7936=1.7936+1$$

$\therefore x=\dfrac{26-4\sqrt{22}}{9}$ is the only solution.

Students will be exposed to application problems where the equation may contain a radical that is a square root. They will solve for the unknown variable by squaring both sides of the equation. Students will then interpret the result in terms of the context.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Solve the quadratic equation: $2x^2 – 5x – 3 = 0$.
- What conditions must be true for a quadratic to have no x-intercepts?
- Determine the radius of a circle that has an area of 60 cm^2. [This is a good question to lead into discussions about assumptions that the radius is positive and inadmissible roots.]

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Solve for *x.*
  (a) $4\sqrt{x+1}-5=7$
  (b) $\sqrt{3x+4}=\sqrt{2x+1}+5$
  (c) $\sqrt{x+4}-2\sqrt{x-1}+1=0$

- Explain why the domain of $y=\sqrt{2x-5}$ is $x\geq\frac{5}{2}$ while the domain of $y=\frac{1}{\sqrt{2x-5}}$ is $x>\frac{5}{2}$. Illustrate your answers graphically with technology.

- State the restrictions for each of the following equations. Next, solve and state any solution(s).
  (a) $\sqrt{4x}=8$
  (b) $\sqrt{x+4}=5$
  (c) $\sqrt{2x-3}=-2$

- The following steps show how a student solved the equation $3+2\sqrt{n+4}=9$.

  $3+2\sqrt{n+4}=9$
  $5\sqrt{n+4}=9$
  $25(n+4)=81$
  $n+4=56$
  $n=52$

  – Is the final answer correct?
  – Find the error(s) in the solution.

- Solve each of the following equations. Check for extraneous roots.
  (a) $\sqrt{3x-5}-2=3$
  (b) $\sqrt{x^2-8x}=3$
  (c) $\sqrt{2x+4x}=3-\sqrt{2x}$

- The time period *T* (in seconds) is the time it takes a pendulum to make one complete swing back and forth. This is modelled by $T=2\pi\sqrt{\frac{L}{32}}$ where *L* is the length of the pendulum in feet. Determine the period of the pendulum if its length is 2 ft.

- The radius of a cylinder can be found using the equation $r=\sqrt{\frac{V}{\pi h}}$ where *r* is the radius, *V* is the volume, and *h* is the height. A cylindrical tank can hold 105.62 m^3 of water. If the height of the tank is 2 m, what is the radius of its base?

- The surface area (S) of a sphere with radius (r) can be found using the equation $S = 4\pi r^2$.
  - Using the given equation, how could you find the radius of a sphere given its surface area? Write the equation.
  - The surface area of a ball is 426.2 cm^2. What is its radius?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use a context such as the following to introduce the idea of restrictions.
- The period (T) of a pendulum can be approximated by the formula $T = 2\pi\sqrt{\frac{l}{g}}$, where l is the length of the pendulum in metres and g is the gravitational constant. What is the gravitational constant on the moon, knowing that a pendulum of 2 metres has a period of 7.02 seconds?

  (**Note:** You are solving for g in this equation. Physicists actually use only the magnitude of the gravitational acceleration and ignore the negative sign that would indicate the fact that it is a "downward" acceleration. With this, we do not need to worry about the restriction where the radicand is negative, but we will have to consider the restriction where g is not equal to zero because it is a denominator.)

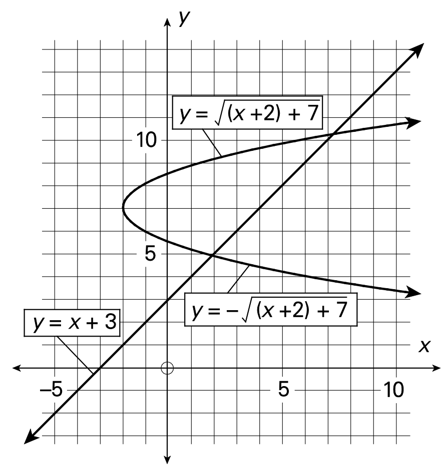
- Before introducing the idea of restrictions in radical expressions, teachers should complete a review of the idea of restrictions with fractions where the denominator cannot be zero, gradually working up to more complicated algebraic expressions so that students will have a strong foundation for future work. For example,

  for $\frac{5}{x}, x \neq 0$; for $\frac{6}{x+5}, x \neq -5$; for $\frac{7}{3x-4}, x \neq \frac{4}{3}$.

- When moving to restrictions in radical expressions, again progress from simpler to more complicated algebraic expressions in which the radicand cannot be negative. For example,

  for $\sqrt{x}, x \geq 0$; for $\sqrt{x-2}, x \geq 2$; for $\sqrt{5x+17}, x \geq -3\frac{2}{5}$; for $\sqrt{(-2x-19)}, x \leq -9\frac{1}{2}$.

- Use technology to clarify the concept of extraneous roots. For example, for $x+3=\sqrt{x+2}+7$, two roots are found x = {2, 7}, but validating shows that 2 is an extraneous root. Graphing $y=x+3$ and $y=\sqrt{x+2}+7$ shows the intersection at x = 7. An intersection at x = 2 can be found by graphing $y=x+3$ and $y=-\sqrt{x+2}+7$. However, in the original equation, we were looking for the principle square root, so only the answer to the positive equation $y=\sqrt{x+2}+7$ is valid.



- Ensure that students are challenged with a variety of questions including solving for a variable in an equation with only one radical, solving for a variable in an equation with two radicals, and solving word problems involving a radical.

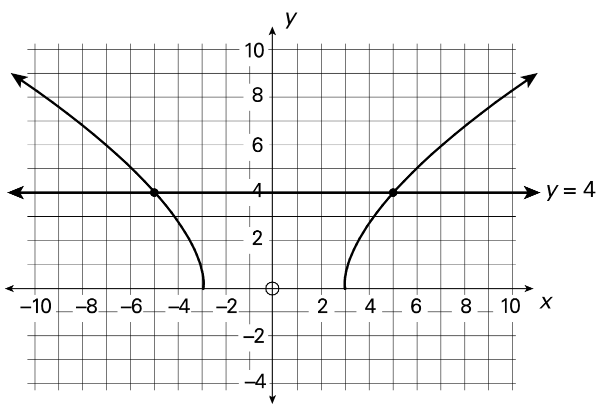
- Rather than squaring both sides of a radical equation, students sometimes mistakenly square the individual terms. When solving $3+\sqrt{2x+1}=7$ for example, they may not isolate the radical. Squaring each term results in the incorrect equation: $3^2+\left(\sqrt{2x+1}\right)^2=7^2$.

  The following illustration could be used to reinforce why squaring individual terms of an equation is not the same as squaring both sides of the equation.

$3 + 4 = 7$	$3 + 4 = 7$
$3^2 + 4^2 = 7^2$	$(3 + 4)^2 = 7^2$
$9 + 16 = 49$	$7^2 = 7^2$
$25 \neq 49$	$49 = 49$

- Use an example, such as $\sqrt{x^2-9}=4$, and the following questions to promote student discussion.
  - What type of equation will this radical equation result in?
  - How is the radicand similar yet different to those studied to date?
  - What values of *x* would result in a radicand that is negative?
  - How many solutions might this equation have?

- Using technology, present the graph of the corresponding functions $y=\sqrt{x^2-9}$ and $y = 4$ for students to analyze.



- From the graph, students should observe that the domain for the function is $x \geq 3$ or $x \leq -3$. This will help them determine whether $x = \pm 5$ are the solutions to the equation.

- Provide students with the following example and ask them to answer the questions.

  - Collision investigators can approximate the initial velocity (*v*) in kilometres per hour, of a car based on the length (l) in metres of the skid mark. The formula $v=12.6\sqrt{l}+8$, where $l \geq 0$ models the relationship.
    - > What length of skid is expected if a car is travelling 50 kmh when the brakes are applied?
    - > How is knowledge of radical equations used to solve this problem?

- Use technology to illustrate the solutions (both extraneous and actual) to equations such as $\sqrt{2x+1}=x-1$.

  Squaring both sides of this equation yields the following:

$$\left(\sqrt{2x+1}\right)^2=(x-1)^2$$
$$2x+1=x^2-2x+1$$
$$0=x^2-4x$$
$$0=x(x-4)$$
$$x=0;\ x=4$$

Check *x* = 0.

$$\sqrt{2x+1}=x-1$$
$$\sqrt{2(0)+1}=0-1$$
$$\sqrt{1}\neq -1$$

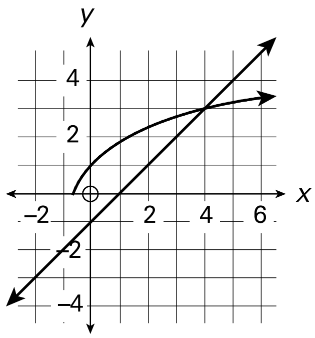
$\therefore x=0$ is extraneous

Check *x* = 4.

$$\sqrt{2x+1}=x-1$$
$$\sqrt{2(4)+1}=4-1$$
$$\sqrt{9}=3$$

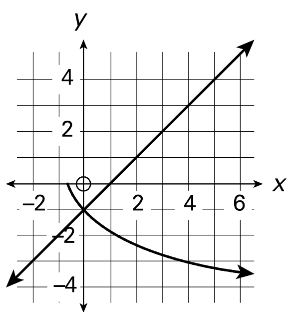
$\therefore x=4$ is the only solution to the equation.

Use technology to graph the equations $y = \sqrt{2x+1}$ and $y = x - 1$.

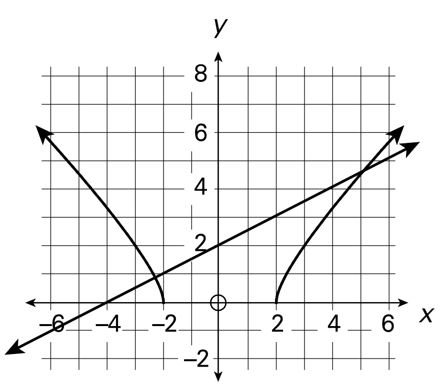


The graph illustrates that the answer $x = 4$ is a solution for $\sqrt{2x+1} = x - 1$.
The extraneous answer ($x = 0$) is the solution to a different equation, $-\sqrt{2x+1} = x - 1$.

The following graph illustrates that the answer $x = 0$ is a solution for $-\sqrt{2x+1} = x - 1$.

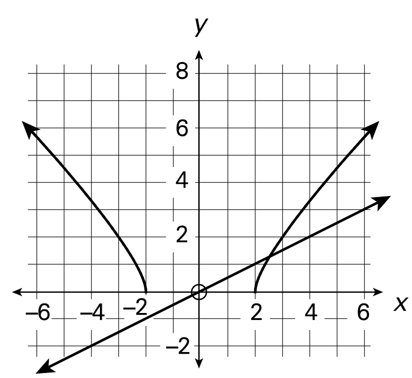


- Students are solving an equation such as $\sqrt{x^2 - 4} = ax + b$. Various possibilities can be illustrated using technology. For example:



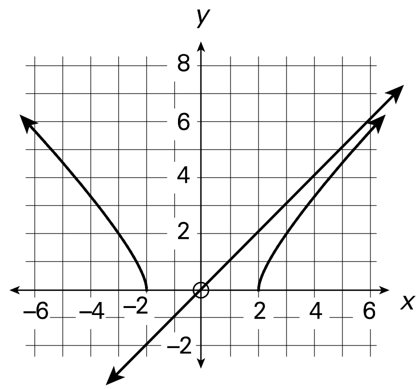
$\sqrt{x^2 - 4} = \frac{1}{2}x + 2$

There are two solutions.



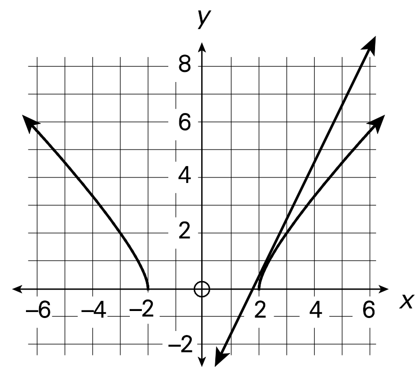
$\sqrt{x^2 - 4} = \frac{1}{2}x$

There is one solution.



$\sqrt{x^2 - 4} = x$

There are no solutions.



$\sqrt{x^2 - 4} = 2x - 2\sqrt{3}$

There is one solution.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper
- metre stick

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- extraneous solution
- inadmissible solution
- radical equation

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 5.3, pp. 294–303

Notes

SCO AN04 Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). [C, ME, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

AN04.01 Compare the strategies for writing equivalent forms of rational expressions to the strategies for writing equivalent forms of rational numbers.
AN04.02 Explain why a given value is non-permissible for a given rational expression.
AN04.03 Determine the non-permissible values for a rational expression.
AN04.04 Determine a rational expression that is equivalent to a given rational expression by multiplying the numerator and denominator by the same factor (limited to a monomial or a binomial), and state the non-permissible values of the equivalent rational expression.
AN04.05 Simplify a rational expression.
AN04.06 Explain why the non-permissible values of a given rational expression and its simplified form are the same.
AN04.07 Identify and correct errors in a simplification of a rational expression, and explain the reasoning.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
AN04 Students will be expected to demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials, and trinomials), concretely, pictorially, and symbolically. (M10)* **AN05** Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**AN04** Students will be expected to determine equivalent forms of rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials).	**RF14** Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials).

* M10—Mathematics 10
** M11—Mathematics 11

Background

In Mathematics 9, students solved problems that involved arithmetic operations on rational numbers (9N03). They will now be introduced to rational expressions limited to numerators and denominators that are monomials, binomials, and trinomials. They will simplify them and determine the non-permissible values.

In this unit, students will simplify a rational expression and determine the non-permissible values. They will perform the operations on rational expressions (addition, subtraction, multiplication, division).

A **polynomial expression**

- can have constants, variables, and exponents
- has exponents that can only be whole numbers
- must contain a finite number of terms
- must not have a variable in the denominator

Therefore, $3x - 4$, $7\sqrt{5}$, $3.5x^4 - \sqrt{2y}$, $2xy - \pi x$, and $5x^2 + 3x - 4$ are polynomial expressions while $5x^2 + 3x^{1.5}$ and $5x^{-2} + 3xy$ are not since they contain a term that has an exponent that is not a whole number.

A **rational expression** is any expression that can be written as the quotient of two polynomials.

Rational expressions can therefore be written in the form $\frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$. To begin work with rational expressions, provide students with several examples of expressions, such as $\frac{2x}{5}$, $\frac{x^2-4}{x+1}$, $\frac{\sqrt{5}}{2}$, $\frac{\sqrt{x}}{2}$, $\frac{x-3}{x-y}$, $\frac{2x-1}{\sqrt[3]{x-4}}$ and ask them to identify and explain why an expression is or is not a rational expression.

Algebraic Fraction	Rational Expression (YES or NO)	Explanation
$\frac{x^2-4}{x+3}$	YES	Ratio of two polynomials
$\frac{x^2-4}{\sqrt{x}+3}$	NO	The denominator is not a polynomial
$\frac{1}{x+3}$	YES	Ratio of two polynomials
$\frac{2\sin x-5}{x}$	NO	The numerator is not a polynomial
$\frac{x^2-y^2}{x+y}$	YES	Ratio of two polynomials

It should be pointed out to students that all rational expressions are algebraic fractions but not all algebraic fractions are rational expressions.

Values of a variable that make the denominator of a rational expression equal zero are not permitted; these are called **non-permissible values**. In Mathematics 7, students were introduced to the concept of why a number cannot be divided by zero (N01). Students should first find the non-permissible values of a rational expression in which the denominator is a first-degree polynomial and then progress to second-degree polynomials.

Students should be exposed to rational expressions with more than one variable. If the expression $2x – 3y$ is in the denominator, for example, the non-permissible values can be found by solving the equation $2x – 3y = 0$ for x or for y. Students always have this option unless it is specifically stated in the problem to solve for a particular variable.

It is important for students to differentiate between non-permissible values (creating zero in the denominator of a fraction) and inadmissible values (those that do not make sense in the context of a problem).

Remind students that **inadmissible values** were discussed in the Quadratics Unit, Mathematics 11 (RF02) and are discussed in Pre-calculus 11 (RF05), when they use quadratic functions to model situations.

Students will continue to work with inadmissible values for a variable in a rational expression.
For example,

- If a boat travelled 20 km with a speed of x kmh, for example, the time taken for the trip would be represented by $\frac{20}{x}$. If students are asked to determine the slowest speed the boat can travel, they should recognize that the non-permissible value is 0 but the inadmissible value is $x < 0$.

In Mathematics 7, students developed skills in writing equivalent positive rational numbers (N07). Students will apply these strategies to rational expressions. This concept is essential when adding and subtracting rational expressions later in this unit.

Students should recognize that they can multiply or divide a rational expression by 1 without changing its value. A rational expression is not equivalent to another rational expression for all values of the independent variable if their restrictions are different.

Consider multiplying by a form of one with no restrictions. $\left(\frac{x-2}{x-5}\right)\bullet\left(\frac{3}{3}\right)=\left(\frac{3x-6}{3x-15}\right)$	Consider multiplying by a form of one with restrictions. $\left(\frac{x-2}{x-5}\right)\bullet\left(\frac{x}{x}\right)=\left(\frac{x^2-2x}{x^2-5x}\right)$
The original expression $\left(\frac{x-2}{x-5}\right)$ existed for all values $x \neq 5$, and the expression $\left(\frac{3x-6}{3x-15}\right)$ also existed for all values $x \neq 5$. Therefore the expressions $\left(\frac{x-2}{x-5}\right)$ and $\left(\frac{3x-6}{3x-15}\right)$ are equivalent.	The original expression $\left(\frac{x-2}{x-5}\right)$ existed for all values $x \neq 5$, and the expression $\left(\frac{x^2-2x}{x^2-5x}\right)$ existed for all values $x \neq 0$ and $x \neq 5$. Therefore, the expressions $\left(\frac{x-2}{x-5}\right)$ and $\left(\frac{x^2-2x}{x^2-5x}\right)$ are equivalent except at the value of $x = 0$.

In Pre-calculus 12, students will learn about points of discontinuity so that they can understand that functions such as $y=\frac{x^2-9}{x-3}$ and $y = x + 3$, are equivalent except when $x = 3$. Their graphs look the same as well except that in the function $y=\frac{x^2-9}{x-3}$ there is a point of discontinuity at (3, 6).

Although graphing rational functions is not an outcome in this course, teachers could illustrate when rational expressions are equivalent for all permissible values of the variable by showing students the graphs of these functions. (Caution should be used when graphing rational functions with a vertical asymptote since some graphing technology will connect points that should not be connected.)

Simplifying a rational expression to simplest terms mirrors the process of simplifying fractions. In both cases, common factors in the numerator and denominator form a ratio of one and can be simplified. Discuss with students the benefit of simplifying rational expressions, whether it is for evaluating or performing operations.

In Mathematics 10 (AN05) and Mathematics 11 (RF02), students factored polynomials. To express rational expressions in equivalent forms, students will build on their understanding of operations on polynomials as well as make use of their factoring skills of polynomials to simplify rational expressions.

Students should recognize that one of the benefits of simplifying an expression is to create an expression that is easier to evaluate. Ask students why the domain of a rational expression is always determined before the expression is simplified. A simplified rational expression, for example, may not have any non-permissible values. However, the simplified expression must retain the non-permissible values of the original expression for both to be equivalent.

It is beneficial to have students analyze solutions that contain errors. Along with providing the correct solutions, students should be able to identify incorrect solutions, including why errors might have occurred and how they can be corrected.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Factor the following:
  (a) $2x^2 - 4x$
  (b) $4x^2 - 9y^2$
  (c) $x^2 - 10x + 25$
  (d) $2x^2 + 5x - 3$

- Explain why it is not possible to divide by zero.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Are the following polynomials?
  (a) $x + 7$
  (b) $\sqrt{x+7}$
  (c) $\sqrt{7x}$

- State the non-permissible values and simplify the following rational expressions.

  (a) $\dfrac{-2m^3n}{6m^2n^4}$ (c) $\dfrac{2x^2+5x-7}{x^2-2x-3}$ (e) $\dfrac{12x^2-54x+24}{3x^2-3x-36}$

  (b) $\dfrac{5x-20}{12-3x}$ (d) $\dfrac{x^2-36}{3x^2-16x-12}$

- Violet made a mistake while simplifying the following rational expression. Determine the error(s) and provide the correct solution:

$$\frac{4x^2+8x-140}{25-x^2}=\frac{4(x^2+2x-35)}{25-x^2}=\frac{4(x+7)(x-5)}{(5+x)(5-x)}=\frac{4(x+7)}{5+x}$$

- Write a rational expression for the following non-permissible values of 0, –2 and 3. Compare your answers with the other members of your class.
- Explain why $x = 2$ is a non-permissible value for $\dfrac{3x}{x-2}$.
- What are the non-permissible values for $\dfrac{x+3}{x^2-16}$?
  - Carmella thinks the non-permissible value is 4.
  - Samuel thinks the non-permissible values are –4 and 4.
  - Ruben thinks the non-permissible values are –4, –3 and 4.

  Who is correct? Justify your answer by solving the problem

Complete the following table.

Are the expressions equivalent?	Yes	No	Justify your choice.
$\frac{x+3}{x-4}$ and $\frac{4x+12}{4x-16}$			
$\frac{5}{x-5}$ and $\frac{5x+25}{x^2-25}$			
$\frac{x+2}{x-3}$ and $\frac{3x+6}{2x-6}$			

- Emile thinks that the expressions $\frac{(x-3)(x+1)}{2x(x+1)}$, $x \neq 0$, $x \neq -1$, and $\frac{x-3}{2x}$, $x \neq 0$ are equivalent. Is Emile right? Explain your reasoning.

- Your cousin says that the expressions $\frac{x-3}{2x}$ and $\frac{(x-3)(x^2+1)}{2x(x^2+1)}$ are equivalent for all values for which they are defined. Is your cousin correct? Explain your reasoning.

- Anja simplifies $\frac{x^2-5x+6}{2x^2-6x}$ as shown below and writes her answer as $\frac{x-2}{2x}$, $x \neq 0$. Is she correct? Explain.

  $\frac{x^2-5x+6}{2x^2-6x}$

  $\frac{(x-3)(x-2)}{2x(x-3)}$

  $\frac{x-2}{2x}, x \neq 0$

- To determine the non-permissible values for the expression $\frac{x-1}{3x^2-12}$, Emile decides to determine what values of *x* make the quadratic $3x^2-12$ equal zero. To do this he decides to factor a 3 out of this quadratic, apply the zero product property, and then use either the quadratic formula or the square root property. Which method is more efficient?

- Identify and correct any errors in the simplification shown below.

  $\frac{8x-12}{6x^2-4x}, x \neq 0, \frac{2}{3}$

  $\frac{4(2x-3)}{2x(3x-2)}$

  $\frac{4}{2x}(1)$

  2x, $x \neq 0, \frac{2}{3}$

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Review equivalent fractions and relate these skills to writing equivalent rational expressions. Begin with multiplying both numerator and denominator by the same factor, working up to multiplying both numerator and denominator by the same algebraic expression. For example,

  (a) $\left(\frac{1}{3}\right) \bullet \left(\frac{2}{2}\right) = \frac{2}{6}$

  (b) $\left(\frac{x+5}{3}\right) \bullet \left(\frac{2}{2}\right) = \frac{2x+10}{6}$

  (c) $\left(\frac{x}{3}\right) \bullet \left(\frac{x+1}{x+1}\right) = \frac{x^2+x}{3x+1}, x \neq -\frac{1}{3}$

  (d) $\left(\frac{x}{3}\right) \bullet \left(\frac{x^2+1}{x^2+1}\right) = \frac{x^3+x}{3x^2+1}$

  (e) $\left(\frac{2}{3x}\right) \bullet \left(\frac{x^2-1}{x^2-1}\right) = \frac{2x^2-2}{3x^3-3x}, x \neq \pm 1, x \neq 0$

  – In each case discuss whether the resulting rational function is equivalent to the original function.
  – Discuss under what conditions it would be necessary to add a restriction to the resulting rational expression.

- Review factoring of polynomials as a basis for learning to simplify rational expressions and to determining non-permissible values for variables, so that the denominator will not equal zero.

  For example, for the expression $\frac{4x+12}{2x^2+7x+3}$,
  - factor $4x + 12$ and $2x^2 + 7x + 3$
  - use factoring to simplify and determine non-permissible values
    $\frac{4x+12}{2x^2+7x+3}=\frac{4(x+3)}{(2x+1)(x+3)}=\frac{4}{2x+1}, x\neq-\frac{1}{2}, x\neq-3$

  Note: both $-\frac{1}{2}$ and –3 are non-permissible even though –3 gives a rational answer in the simplified expression $\frac{4}{2x+1}$.

- Given an expression such as $\frac{x}{x+2}$, use the following prompts to promote student discussion around non-permissible values:
  - Using inspection, what value of x would make the denominator zero?
  - Explain why this value of x is called a non-permissible value.

- Fill in the following table of values. What do you notice? What is the domain of the expression $\frac{x}{x+2}$?

x	–3	–2	–1	0	1	2	3
y							

  Students should notice that the non-permissible value of $\frac{x}{x+2}$ is –2.

  This can be written as $\frac{x}{x+2},\ x\neq-2$.

- Discuss an expression, such as $\frac{x-1}{3x^2-12}$. Ask students to answer the following questions:
  - Using inspection, what value(s) of x would make the denominator zero?
  - What other strategies can be used to solve the quadratic equation?
  - Are there any rational expressions without non-permissible values?

- When determining the restricted values for an expression such as $\frac{x-1}{3x^2-12}$, students will solve $3x^2 - 12x = 0$. In cases such as these students may remove the greatest common factor (GCF) and apply the zero product property. In addition to incorrectly factoring the equation, some common errors are as follows:
  - If students divide the equation $3x^2 - 12x = 0$ by $3x$ to obtain $x - 4 = 0$ and conclude that $x = 4$. The problem with this approach is that when they divide by x they have introduced the restriction $x\neq0$, but when checked, one of the solutions to the equation $3x^2 - 12x = 0$ is zero.
  - Another error occurs when students factor $3x^2 - 12x = 0$ as $3x(x - 2)(x + 2)$ and include 3 as a non-permissible value.

- The idea that a rational expression is not equivalent to another rational expression if their restrictions are different can be confusing. You will need to take time to develop this idea.
  - Consider the rational expression $\frac{4}{x}$ where $x \neq 0$. Ask students to write a rational expression by multiplying both the numerator and denominator by 2, by x, and by $x + 1$. Did any of their expressions produce a new restriction?
  - Ask students to verify if the expressions are equivalent by using substitution. When the expressions $\frac{4}{x}$, $x \neq 0$ and $\frac{4(x+1)}{x(x+1)}$, $x \neq 0$, $x \neq -1$ are compared, both are undefined at $x = 0$.

    When $x = -1$, however, the expression $\frac{4}{x}$ simplifies to –4 while the expression $\frac{4(x+1)}{x(x+1)}$ is undefined. Since the expressions are not equal for the same value of x, the expressions are not equivalent unless you restrict their domains.
  - Therefore, you can correctly state that $\frac{4}{x} = \frac{4(x+1)}{x(x+1)}$, $x \neq 0$, $x \neq -1$ or $\frac{4}{x} = \frac{4(x+1)}{x(x+1)}$, $x \neq -1$ but it would not be correct to state that $\frac{4}{x} = \frac{4(x+1)}{x(x+1)}$ or that $\frac{4}{x} = \frac{4(x+1)}{x(x+1)}$, $x \neq 0$.

- When writing an equivalent expression, remind students to use the distributive property appropriately. When simplifying $\left(\frac{x}{x+4}\right) \bullet \left(\frac{2}{2}\right)$, for example, students may incorrectly write $\frac{2x}{2x+4}$ or $\frac{2x}{x+8}$. To avoid this error, encourage students to place brackets around the binomial $\frac{2(x)}{2(x+4)}$ when multiplying.

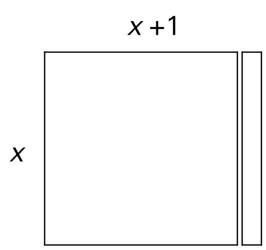
- To emphasize the benefit of simplifying rational expressions, whether for evaluating or performing operations. Ask students to evaluate the expression $\left(\frac{x^2+4x}{x}\right)$ at $x = 2$ and then to simplify the expression $\left(\frac{x^2+4x}{x}\right)$ to obtain $(x + 4)$, $x \neq 0$ and evaluate that expression at $x = 2$.

  Students should answer the following questions:
  - What is the result when substituting the value into the original expression?
  - What is the result when substituting the value into the simplified expression?
  - Why were the results the same?
  - What is the benefit of simplifying an expression before substituting values for the variables?
  - Why does the simplified expression include a non-permissible value?

- A common error occurs when students assume that the non-permissible value for x is zero rather than looking at the value(s) of x that produce a denominator of zero. Encourage them to substitute the non-permissible value(s) for x back into the denominator to verify the denominator results in zero.

- When simplifying rational expressions, students may incorrectly "cross out" terms rather than divide factors. They may simplify, for example, $\frac{x^2+x}{x^2-1}$ as $\frac{\cancel{x^2}+x}{\cancel{x^2}-1}$ resulting in $-x$. To help students see this error, ask them to make a comparison with a numerical rational expression such as $\frac{8}{12}=\frac{5+3}{5+7}$ and $\frac{8}{12}=\frac{\cancel{5}+3}{\cancel{5}+7}\neq\frac{3}{7}$. Ask them if $\frac{8}{12}$ is equal to $\frac{3}{7}$. Students should understand that "crossing out" a portion of the factor is incorrect. Another error occurs when students omit a numerator of 1 after the rational expression is simplified. They mistakenly simplify $\frac{3}{6x}$, for example, as $2x$. Encourage students to check how reasonable their answers are by rewriting the expression as $\frac{3}{6}\bullet\frac{1}{x}$.

- Algebraic errors can be avoided by emphasizing understanding rather than procedure. If students are making errors simplifying expressions such as $\frac{x^2+x}{x^2-1}=\frac{\cancel{x^2}+x}{\cancel{x^2}-1}=-x$ it indicates that they do not understand the meaning of the algebraic expressions. Consider revising alge-tiles for the question $\left(\frac{x^2+x}{x+1}\right)$. To illustrate this, the area of the rectangle would be (x^2+x) while one of its dimensions would be ($x+1$).



- Some possible activities that reinforce working with rational expressions are described below.
  - As students come into class, hand each of them a rational expression written on a card, and have them find a classmate with a rational expression with at least one identical non-permissible value.
  - Ask students to work in groups to participate in the Domino Game. Provide each group with 10 domino cards. One side of the card should contain a rational expression, while the other side contains non-permissible values for a different rational expression. The task is for students to lay the dominoes out such that the non-permissible values on one card will match with the correct rational expression on another. Students will eventually form a complete loop with the first domino card matching with the last card. A sample is shown below:

$\frac{3x}{2x-1}$	$x\neq\frac{1}{3}$	$\frac{x+2}{3x-1}$	$x\neq-\frac{1}{2}$	$\frac{7x}{2x+1}$	$x\neq 2$

  - Divide the class into two groups. One group will be given rational expressions and the other group will be given the associated rational expression in simplest form. Ask students to find a partner who has a rational expression equivalent to theirs.

SUGGESTED MODELS AND MANIPULATIVES

- algebra tiles

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- inadmissible values
- non-permissible values
- polynomial
- radical expression

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 6.1, pp. 310–321

Notes

SCO AN05 Students will be expected to perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials). [CN, ME, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

AN05.01 Compare the strategies for performing a given operation on rational expressions to the strategies for performing the same operation on rational numbers.
AN05.02 Determine the non-permissible values when performing operations on rational expressions.
AN05.03 Determine, in simplified form, the sum or difference of rational expressions with the same denominator.
AN05.04 Determine, in simplified form, the sum or difference of rational expressions in which the denominators are not the same and which may or may not contain common factors.
AN05.05 Determine, in simplified form, the product or quotient of rational expressions.
AN05.06 Simplify an expression that involves two or more operations on rational expressions.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
AN04 Students will be expected to demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials, and trinomials), concretely, pictorially, and symbolically. (M10) **AN05** Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**AN05** Students will be expected to perform operations on rational expressions (limited to numerators and denominators that are monomials, binomials, or trinomials).	**RF14** Students will be expected to Graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials).

* M10—Mathematics 10
** M11—Mathematics 11

Background

In Mathematics 9, students solved problems involving operations on rational numbers (9N03). This will now be extended to adding, subtracting, multiplying and dividing rational expressions with numerators and denominators limited to monomials, binomials, and trinomials.

Multiplying and dividing rational expressions is very similar to the process students used to multiply and divide rational numbers. Using examples such as $\frac{12}{10} \bullet \frac{10}{21}$ and $\frac{x^2-9}{x^2-4x} \bullet \frac{x-4}{x-3}$, ask students to simplify and find the product for each. Students should think about whether the strategy for multiplying rational expressions is the same as the strategy for multiplying rational numbers. Ask them to also consider at what step the non-permissible values are determined.

Non-permissible values will be determined for denominators of rational expressions, ensuring that denominators do not equal zero. In the case of division, all expressions within the divisor must also be considered when stating non-permissible values. For example,

$$\frac{2x-4}{x^2+9x+20} \div \frac{x^2+x-6}{x^2+7x+12} = \frac{2(x-2)}{(x+5)(x+4)} \times \frac{(x+4)(x+3)}{(x+3)(x-2)} = \frac{2}{x+5} \qquad x \neq -5, x \neq -4, x \neq -3.2$$

It is important for students to recognize the importance of factoring the numerator and denominator of the rational expression, if possible, before the product is determined.

In Mathematics 10, students were introduced to the least common multiple for a set of numbers (10AN01). Students should compare finding the lowest common denominator of rational numbers to finding the lowest common denominator of rational expressions. Allow students to discover the different situations that occur when finding the lowest common denominator of two numbers and then compare this to rational expressions. Consider the following table:

Rational Number	**Situation**	**Rational Expression**
$\frac{3}{7}-\frac{2}{7}$	The denominators are the same.	$\frac{x^2}{x+1}-\frac{1}{x+1}, x \neq -1$
$\frac{1}{12}+\frac{5}{6}$	One denominator is a multiple of the other.	$\frac{3}{x+5}-\frac{1}{4x+20}, x \neq -5$
$\frac{2}{3}+\frac{7}{2}$	The denominators have no common factors.	$\frac{3}{2x}+\frac{4}{x-1}, x \neq 1, x \neq 0$
$\frac{5}{14}-\frac{1}{6}$	The denominators have a common factor.	$\frac{7}{x^2-9}+\frac{1}{4x+12}, x \neq \pm 3$

Similar to rational numbers, rational expressions can be added if they have common denominators. Once students determine the lowest common denominator, they should rewrite each rational expression with that common denominator.

A **complex fraction** is an example of an expression involving two or more operations on a rational expression. In order to avoid errors, students should place brackets appropriately and use the order of operations correctly.

It is important to provide examples to students illustrating various strategies to simplify an expression containing a complex fraction.

Method A: Students may first simplify both the numerator and denominator, inverting, multiplying, and then simplifying.

Consider the following example:

$$\frac{\frac{1}{x+2}+\frac{1}{x-2}}{\frac{x}{x^2-4}+\frac{1}{x+2}}=\frac{\frac{x-2}{(x+2)(x-2)}+\frac{x+2}{(x+2)(x-2)}}{\frac{x}{(x+2)(x-2)}+\frac{x-2}{(x+2)(x-2)}}=\frac{2x}{(x+2)(x-2)}\times\frac{(x+2)(x-2)}{2x-2}=\frac{2x}{2x-2}=\frac{x}{x-1}$$

Remember that the restrictions $x \neq \pm 2, x \neq 1$ must be stated in order to be able to confirm that these rational expressions are equivalent.

Method B: Students can multiply the entire expression by the common denominator divided by itself. This common denominator is obtained by considering all existing denominators in the expression.

$$\left(\frac{\frac{1}{x+2}+\frac{1}{x-2}}{\frac{x}{x^2-4}+\frac{1}{x+2}}\right)\frac{(x+2)(x-2)}{(x+2)(x-2)}=\frac{\frac{1}{x+2}(x+2)(x-2)+\frac{1}{x-2}(x+2)(x-2)}{\frac{x}{(x+2)(x-2)}(x+2)(x-2)+\frac{1}{x+2}(x+2)(x-2)}=\frac{x-2+x+2}{x+x-2}=\frac{2x}{2x-2}=\frac{x}{x-1}$$

Remember that the restrictions $x \neq \pm 2, x \neq 1$ must be stated in order to be able to confirm that these rational expressions are equivalent.

As students work through the two possible strategies, they should think about efficiency and what will work best for any complex fraction.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Perform the indicated operations and simplify the results.
  (a) $\frac{4}{5}+\frac{2}{3}$
  (b) $\frac{5}{12}\bullet\frac{6}{35}$
  (c) $\frac{12}{17}\div\frac{3}{34}$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Simplify the following rational expressions and determine the non-permissible values.
  (a) $\frac{5}{4m}+\frac{1}{3m}$
  (b) $\frac{6}{x+5}+\frac{x}{x+5}$
  (c) $\frac{2}{(x-2)(x+3)}-\frac{5}{x+3}$
  (d) $\frac{x-1}{x^2-x-2}-\frac{2x+4}{x^2-7x+10}$
  (e) $\frac{3t^2}{-2y}\div\frac{6t^3}{4y^2}$
  (f) $\frac{x^2+7x}{x^2-1}\times\frac{x^2+3x+2}{x^2+14x+49}$
  (g) $\frac{x^2+4x+4}{4-x^2}\times\frac{2x-4}{x^2-3x-10}\div\frac{4x+16}{x^2-25}$
  (h) $\frac{x-3}{x+2}+\frac{8x+4}{x^2+2x-35}\times\frac{x^2+4x-21}{2x^2-5x-3}$

- Find the product of $\left(\frac{x^2}{x^2-4}\right)\bullet\left(\frac{x+2}{x}\right)$ using two different strategies. Which strategy is more efficient? Explain.

- Work in pairs to complete the following table. Explain the similarities between finding the lowest common denominator of two rational numbers as opposed to finding the LCM of two rational expressions.

Rational Number	LCM	Rational Expression	LCM	Similarities
$\frac{4}{5}+\frac{3}{5}$		$\frac{6}{2x-1}+\frac{2}{2x-1},\ x\neq\frac{1}{2}$		
$\frac{1}{5}-\frac{7}{15}$		$\frac{4x}{x-3}-\frac{5}{6x-18},\ x\neq 3$		
$\frac{7}{12}+\frac{3}{8}$		$\frac{2}{x^2-36}+\frac{4}{3x-18},\ x\neq\pm 6$		
$\frac{1}{3}+\frac{2}{5}$		$\frac{2}{x-3y}-\frac{4}{x+5y},\ x\neq 3y,\ x\neq -5y$		

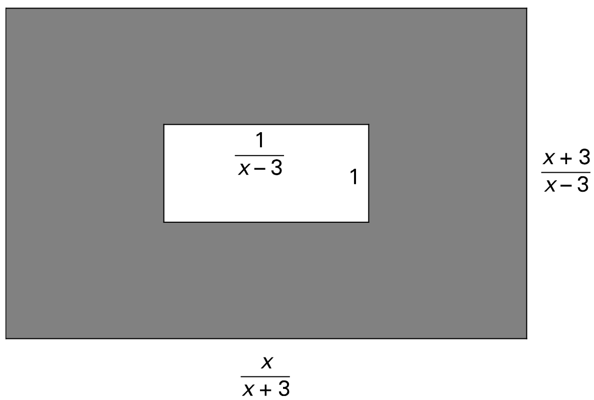
- Liban stated that the permissible values for the quotient and the product of the expressions $\frac{x^2+4x+3}{x^2-16}$ and $\frac{x^2-2x+8}{x^2-7x+12}$ are the same. Do you agree or disagree with his statement? Justify your answer.

- Simplify the rational expressions. Remember that part of your solution must be a statement of any restrictions.

  (a) $\frac{x+7}{2x+14}-\frac{5x}{-3x-21}$

  (b) $\frac{2x-6}{x^2-x-6}-\frac{3x+12}{x^2+x-12}$

- Find the area of the shaded region.



- Simplify the following expressions. Make sure to state any restrictions.

  (a) $\frac{2(2x+1)}{x^2+x-6}-\frac{2}{x+3}$

  (b) $\frac{x}{x-y}+\frac{x^2+y^2}{x^2-y^2}+\frac{y}{x+y}$

  (c) $\left(\frac{x^2+4x+4}{x^2-8x+15}\right)\div\left(\frac{4x^2+8x}{x^2-9}\right)$

  (d) $\dfrac{4-\frac{12}{x+2}}{\frac{4}{x-3}+2}$

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Review operations with fractions and the factoring of polynomials to build new knowledge on a strong foundation as students develop skills for simplifying rational expressions.

- Explore with students the connection between finding a common denominator with rational numbers and finding a common denominator with rational expressions.

  Example 1: The two denominators below do not have any common factors; therefore, the LCM will be the product of the two.

  Rational Numbers

  $$\frac{4}{5}+\frac{3}{2}$$
  $$=\frac{8}{10}+\frac{15}{10}$$
  $$=\frac{23}{10}$$

  Rational Expressions

  $$\frac{5}{x+1}+\frac{3}{x+4}$$
  $$=\frac{5}{(x+1)}\frac{(x+4)}{(x+4)}+\frac{3}{(x+4)}\frac{(x+1)}{(x+1)}$$
  $$=\frac{(5x+20)+(3x+3)}{(x+1)(x+4)}$$
  $$=\frac{8x+23}{(x+1)(x+4)} \quad x\neq -1,\ x\neq -4$$

Example 2: The two denominators have a common factor; therefore, the LCM will not be the product of the two denominators, but the product of the common factor and the remaining uncommon factors.

Rational Numbers

LCM = 2 × (3 × 5)

$$\frac{5}{6}+\frac{3}{10}$$
$$=\frac{5}{3\times2}+\frac{3}{5\times2}$$
$$=\frac{5}{3\times2}\times\frac{5}{5}+\frac{3}{5\times2}\times\frac{3}{3}$$
$$=\frac{25}{30}+\frac{9}{30}$$
$$=\frac{34}{30}$$

Rational Expressions

LCM = (x + 2) × [(x – 1)((x – 2)]

$$\frac{5}{x^2+x-2}+\frac{2}{x^2-4}$$
$$=\frac{5}{(x+2)(x-1)}+\frac{2}{(x+2)(x-2)}$$
$$=\frac{5}{(x+2)(x-1)}\frac{(x-2)}{(x-2)}+\frac{2}{(x+2)(x-2)}\frac{(x-1)}{(x-1)}$$
$$=\frac{(5x-10)+(2x-2)}{(x+2)(x-1)(x-2)}$$
$$=\frac{7x-12}{(x+2)(x-1)(x-2)} \qquad x\neq\pm2,\ x\neq1$$

- Reinforce that multiplication of rational expressions follows the same procedure as multiplying rational numbers, but with the added necessity of determining the non-permissible values for the variables.

- Ask students to answer the following questions related to specific rational expressions.
  - How do you find the lowest common denominator?
  - Why is it beneficial to simplify the expression before finding the lowest common denominator?
  - What are the non-permissible values?
  - Can you list other examples that fit each situation?

- A common student error involves adding or subtracting the numerators without first writing the fractions with a common denominator. For example, students mistakenly add $\frac{x}{5}+\frac{2}{3}$ as $\frac{x+2}{8}$.

- Remind students to be careful when subtracting rational expressions. They sometimes forget to distribute the negative sign when there is more than one term in the numerator.

  For example, $\frac{3x-2}{(x+2)(x-2)}-\frac{2x-4}{(x+2)(x-2)}$ is often incorrectly written as $\frac{3x-2-2x-4}{(x+2)(x-2)}$. Encourage students to use parentheses, like $\frac{(3x-2)-(2x-4)}{(x+2)(x-2)}$, to help them avoid this mistake.

- Some possible learning tasks that reinforce working with rational expressions are described below.
  - Ask students to create an activity sheet in which the column on the left contains operations with rational expressions and the column on the right contains the non-permissible values (not in the same order). Students will then exchange their sheets. The task is to match each expression with its correct non-permissible values (the non-permissible values may match more than one expression on the left and may not match any).

- Ask students, working in pairs or small groups, to create two rational expressions. The first rational expression should contain each of the operations. The other expression would involve two or more operations on rational expressions. Students should solve the expressions and present their findings to the class.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- complex fractions
- non-permissible values

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 6.2 and 6.3, pp. 322–340

Notes

SCO AN06 Students will be expected to solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials, or trinomials). [C, PS, R]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(*It is intended that the rational equations be those that can be simplified to linear and quadratic equations*.)

AN06.01 Determine the non-permissible values for the variable in a rational equation.
AN06.02 Determine the solution to a rational equation algebraically, and explain the process used to solve the equation.
AN06.03 Explain why a value obtained in solving a rational equation may not be a solution of the equation.
AN06.04 Solve problems by modelling a situation using a rational equation.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
AN04 Students will be expected to demonstrate an understanding of the multiplication of polynomial expressions (limited to monomials, binomials, and trinomials), concretely, pictorially, and symbolically. (M10)* **AN05** Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. (M10) **RF10** Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**AN06** Students will be expected to solve problems that involve rational equations (limited to numerators and denominators that are monomials, binomials, or trinomials).	**RF14** Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials).

* M10—Mathematics 10
** M11—Mathematics 11

Background

In Mathematics 9, students solved linear equations (PR03). In Mathematics 10 students solved systems of linear equations (RF10). In Mathematics 11, students solved quadratic equations and identified inadmissible roots (RF02).

Students will now solve equations containing rational expressions and check the solutions. It is intended that the rational equations be those that can be simplified to linear and quadratic equations.

Students will be exposed to different strategies when solving rational equations. It would be beneficial to begin with an example that is easier to visualize before moving on to more complex equations. Some students may use trial and error to solve an equation such as $\frac{x}{10}=\frac{2}{5}$. Others may be able to determine the solution by inspection. Encourage students to discuss their ideas. For example, a student may respond that in order to get the number 10, the number 5 must be doubled; therefore, 2 is also doubled resulting in $x = 4$. This student response is a great lead-in to the strategy of creating an equivalent rational equation with common denominators.

Ask students to rewrite the rational equation with a common denominator $\frac{x}{10}=\frac{4}{10}$ and then write an equation with the numerators.

Another strategy involves eliminating the denominators. Use an example such as $\frac{x}{10}=\frac{2}{5}$ to promote discussion around LCM.

Once the strategies have been discussed, students will then solve more complex rational equations.

When solving $\frac{3}{x}+\frac{7}{2x}=\frac{1}{5}$, for example, students may find a common denominator for the left side of the equation $\frac{6}{2x}+\frac{7}{2x}=\frac{1}{5}$ or $\frac{13}{2x}=\frac{1}{5}$ and then proceed to solve for the value of x by either multiplying both sides by $10x$ or by equating the denominators. They can also multiply both sides of the equation by the lowest common denominator prior to combining terms $(10x)\left(\frac{3}{x}\right)+(10x)\left(\frac{7}{2x}\right)=(10x)\left(\frac{1}{5}\right)$ to obtain $\left(\frac{30x}{x}\right)+\left(\frac{70x}{2x}\right)=\left(\frac{10x}{5}\right)$ or $(30) + (35) = (2x)$.

It is important for students to recognize that they can reduce the numbers of steps by multiplying both sides of the original equation by the lowest common denominator.

Students can add or subtract the terms on the left-hand side or the right-hand side of the equation before they multiply to eliminate the denominators. This process, however, may lead to an equation where the degree of the polynomial is greater than what they started with. Consider the following example:

$$\frac{2x^2+1}{x+3}=\frac{x}{4}+\frac{5}{x+3}$$
$$\frac{2x^2+1}{x+3}=\frac{(x+3)(x)+(4)(5)}{4(x+3)}$$
$$\frac{2x^2+1}{x+3}=\frac{x^2+3x+20}{4(x+3)}$$
$$(2x^2+1)(4)(x+3)=(x^2+3x+20)(x+3)$$

This example results in a cubic equation. Students are familiar only with solving quadratic equations at this point. Thus, multiplying both sides of the equation by the lowest common denominator would be the method students would choose.

Alternately, students could obtain two equations with the same denominator. They would then be able to **equate the numerators**. That is,

$$\frac{2x^2+2}{x+2}=\frac{x^2+3x+22}{2x+4};\ x\neq -2$$
$$\frac{2x^2+2}{x+2}=\frac{x^2+3x+22}{2(x+2)}$$
$$\frac{(2)(2x^2+2)}{2(x+2)}=\frac{x^2+3x+22}{2(x+2)}$$
$$\therefore (2)(2x^2+2)=x^2+3x+22$$

When solving rational equations, the modified equation may result in either a linear or quadratic equation. Students will have a choice whether to use the quadratic formula, to complete the square (if done at this point in the course), or to use their factoring skills to solve the quadratic equation. Remind students to verify their solutions to avoid extraneous roots.

Caution students that it is necessary to ensure that they are aware of any non-permissible values and that they check to ensure that any solutions obtained satisfy the equation.

$$(2)(2x^2+2)=x^2+3x+22$$
$$4x^2+4=x^2+3x+22$$
$$3x^2-3x-18=0$$
$$x^2-x-6=0$$
$$(x-3)(x+2)=0$$
$$x-3=0;\ x+2=0$$
$$x=3;\ =-2 \text{ (non-permissible value)}$$
$$\therefore x=3$$

Students should recognize solutions that are non-permissible values are extraneous roots and must be eliminated as a valid solution. In this, $x = 3$ is the only solution.

Rational equations can be solved by multiplying both sides of the equation by a common denominator. This will eliminate all fractions from the equation. The resulting equation can then be solved. However, it is possible that multiplying by a common denominator may result in an answer that was a non-permissible solution. It is therefore necessary for students to always check after they have found possible solutions to a rational equation, as there is no easy way to predict whether possible solutions will work in the original equation. For example,

Equating the denominators

$$\frac{1}{x-2}=\frac{3}{x+2}-\frac{6x}{x^2-4}$$

$$\left(\frac{1}{x-2}\right)\left(\frac{x+2}{x+2}\right)=\left(\frac{3}{x+2}\right)\left(\frac{x-2}{x-2}\right)-\frac{6x}{(x-2)(x+2)}$$

$$\frac{x+2}{(x-2)(x+2)}=\frac{3x-6}{(x-2)(x+2)}-\frac{6x}{(x-2)(x+2)}$$

$$\frac{x+2}{(x-2)(x+2)}=\frac{3x-6-6x}{(x-2)(x+2)}$$

Equating numerators

$$\therefore x+2=-6-3x$$

$$4x=-8$$

$$x=-2$$

Multiplying to eliminate the denominators

$$\frac{1}{x-2}=\frac{3}{x+2}-\frac{6x}{x^2-4}$$

Multiply both sides by (x^2-4) or $(x+2)(x-2)$

$$\left(\frac{1}{x-2}\right)\bullet(x^2-4)=\left(\frac{3}{x+2}\right)\bullet(x^2-4)-\left(\frac{6x}{x^2-4}\right)\bullet(x^2-4)$$

$$\left(\frac{x^2-4}{x-2}\right)=\left[\frac{3(x^2-4)}{x+2}\right]-\left[\frac{6x(x^2-4)}{x^2-4}\right]$$

$$\left[\frac{(x+2)(x-2)}{x-2}\right]=\left[\frac{3(x+2)(x-2)}{x+2}\right]-\left[\frac{6x(x^2-4)}{x^2-4}\right]$$

$$(x+2)=3(x-2)-6x$$

$$x+2=3x-6-6x$$

$$\therefore x+2=-6-3x$$

$$4x=-8$$

$$x=-2$$

However, we can see that $x = -2$ is a non-permissible solution to the original equation. Therefore, this equation has no solution.

Students will be expected to write an equation to represent a problem. They should be exposed to examples such as the following:

The sum of a number and its reciprocal is $\frac{5}{2}$.

Students can begin this example using trial and error and discuss possible solutions. They should then proceed to write the rational equation $x+\frac{1}{x}=\frac{5}{2}$. Encourage students to write their own examples and share with the class.

It is important for students to recognize that inadmissible roots come from the context of the problem.

Discuss different scenarios that produce inadmissible roots. A negative numerical value, for example, would not make sense if referring to time, height, and length.

Students may have difficulty interpreting the information from the word problem and writing the rational equation. Encourage them to use tables and diagrams to help them break down the information.

When solving an equation, encourage students to check that the solutions satisfy the original equation, are permissible, and in the case of a word problem, realistic in the context.

Students should also be exposed to word problems that produce a rational equation resulting in solving a quadratic equation. In such cases, there may be an inadmissible value that will need to be rejected in the context of the problem.

For some word problems, students will need to be familiar with the relationship between distance, speed, and time ($d = vt$). Be sure to equip them with the information they need to solve the problem.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in assessing students' prior knowledge.

- Solve for the variable:
  (a) $\frac{3}{7}=\frac{k}{7}$
  (b) $\frac{4}{9}=\frac{4}{m}$
  (c) $\frac{3}{5}=\frac{p}{7}$
  (d) $\frac{x}{3}+2=7$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Algebraically solve the following equations for all solutions:

  (a) $\frac{x}{x^2-2x+1}=\frac{2}{x+1}+\frac{4}{x^2-1}$

  (b) $\frac{2}{x+2}-\frac{x}{2-x}=\frac{x^2+4}{x^2-4}$

- Reflect on the process of solving a rational equation. Respond in writing to three reflective prompts providing six responses, as shown below, to describe what you have learned. Some suggestions are as follows:
  - The biggest area in which I'm struggling is ... and I will work on it by ...
  - The three most important things I've learned so far are ...
  - The three things I understand best are ...
  - The top two ideas I'm struggling with are ...

- Reflect on Rebekah's solution to the following question:
  - It takes Mike 9 hours longer to construct a fence than it takes Carmella. If they work together, they can construct the fence in 20 hours. How long would it take Mike to construct the fence alone?

    Rebekah solved the equation $\frac{20}{t}+\frac{20}{t+9}=1$ and stated the solutions to the word problem were 36 and –5. Verify the solution and state whether Rebekah is correct.

- Solve the following rational equations:

  (a) $\frac{3}{a+5}=\frac{2}{a+4}$

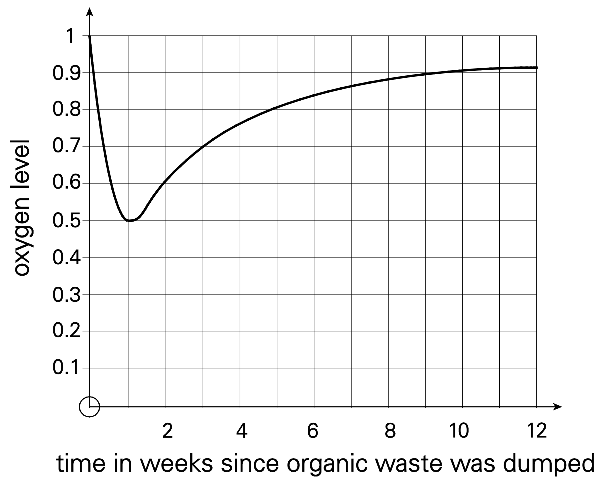
  (b) $y-\frac{3}{2}=\frac{7}{y}$

  (c) $5+\frac{2m}{m^2-m-6}=\frac{m}{2m-6}$

- Given that $\frac{A}{x+3}+\frac{B}{x-2}=\frac{3x-4}{x^2+x-6}$, determine the values of *A* and *B*.

- A family on vacation leaves home and travels 266 km to Halifax, Nova Scotia. The family stops for a quick lunch and continues for 96 km on to their cottage on the Eastern Shore of Nova Scotia along the scenic route at a speed of 15 kmh less than the first part of the trip. If the total driving time took 4 hours, how fast were they travelling on the scenic route?

- Let t be the time in weeks. At time $t = 0$, organic waste is dumped into a pond. The oxygen level in the pond at time t is given by $f(t)=\dfrac{t^2-t+1}{t^2+1}$. Assume that $f(0) = 1$ is the normal level of oxygen. The graph of this function is shown below



(a) What is the contextual significance of the minimum point on this graph?
(b) What eventually happens to the oxygen level?
(c) Approximately how many weeks must pass before the oxygen level returns to 75% of its normal level?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use an example such as $\frac{x+1}{12}=\frac{5}{3}$, to promote discussion around lowest common denominator.
  - What is the least common multiple of 12 and 3?
  - What would happen if the lowest common denominator was multiplied on both sides of the equation? Why is this mathematically correct?
  - What is the simplified equation?
  - What is the solution?

- When solving rational equations, students may wish to "cross multiply" to obtain an equation to solve. Avoid the use of this terminology as it does not emphasize understanding. Instead talk about "multiplying both sides of the equation by a value or expression" or "multiplying a rational expression by a form of one to obtain a common denominator." It is critical that students understand that preservation of equality be maintained.
  For example,

$\frac{2x-1}{3}=\frac{x+1}{2}$	
$6\left(\frac{2x-1}{3}\right)=6\left(\frac{x+1}{2}\right)$	Multiplying both sides of the equation by 6.
$2(2x-1)=3(x+1)$	Simplifying the fractions.
OR	
$\frac{2x-1}{3}=\frac{x+1}{2}$	
$\frac{2}{2}\left(\frac{2x-1}{3}\right)=\frac{3}{3}\left(\frac{x+1}{2}\right)$	Multiplying each expression by a form of 1 to obtain a common denominator.
$\frac{2(2x-1)}{6}=\frac{3(x+1)}{6}$	
$2(2x-1)=3(x+1)$	Equating the numerators.

- Given the equation $\frac{2x+3}{x+5}+\frac{1}{2}=-\frac{14}{2(x+5)}$, ask students to answer the following questions.
  - What is the non-permissible root? What does this mean?
  - What is the solution to the resulting linear equation?
  - Why is it important to check the solution by using the original equation?

- To introduce working with applications involving rational equations, consider the following example.
  - Sherry mows a lawn in 4 hours. Maxwell mows the same lawn in 5 hours. How long would it take both of them working together to mow the lawn?

  Pose the following questions to begin a discussion.
  - How much of the lawn would Sherry mow in 1 hour?
  - How much of the lawn would Maxwell mow in 1 hour?
  - How much of the lawn would both mow together in 1 hour?

Completing a table such as the one below should help students organize their information.

	Time to mow lawn (hours)	**Fraction of lawn mowed in 1 hour**
Sherry	4	$\frac{1}{4}$
Maxwell	5	$\frac{1}{5}$
Both	x	$\frac{1}{x}$

- Many word problems requiring the use of rational equations will deal with questions such as moving with or against current, with a head or tail wind, with an increase or decrease in speed. These situations can be conceptualized with the use of video clips found on sites such as *YouTube* (e.g., Dan Meyer walking up and down an escalator). Be creative in finding clips that will engage student interest.

- The following example illustrates suggested steps to follow when solving a rational equation.

  Step 1: Factor if necessary and list the non-permissible values.

  $$\frac{m}{m+1}-\frac{3m}{m^2-1}=4$$

  Step 2: Multiply each term by the lowest common denominator **OR** equate the denominators.

  Lowest Common Denominator:

  $$\frac{m}{m+1}-\frac{3m}{(m+1)(m-1)}=4;\ m\neq -1,\ m\neq 1 \text{ (non-permissible values)}$$
  $$\frac{m}{m+1}(m+1)(m-1)-\frac{3m}{(m+1)(m-1)}(m+1)(m-1)=4(m+1)(m-1)$$
  $$\frac{(m)(m+1)(m-1)}{m+1}-\frac{3m(m+1)(m-1)}{(m+1)(m-1)}=(4)(m+1)(m-1)$$
  $$\therefore (m)(m+1)-(3\mathrm{m})=(4)(\mathrm{m}+1)(\mathrm{m}-1)$$

  Equating the Denominators:

  $$\frac{m}{m+1}-\frac{3m}{(m+1)(m-1)}=4$$
  $$\left(\frac{m}{m+1}\right)\left(\frac{m-1}{m-1}\right)-\frac{3m}{(m+1)(m-1)}=(4)\left[\frac{(m+1)(m-1)}{(m+1)(m-1)}\right]$$
  $$\frac{m(m-1)-(3m)}{(m+1)(m-1)}=\frac{(4)(m+1)(m-1)}{(m+1)(m-1)}$$
  $$\therefore m(m-1)-(3m)=(4)(m+1)(m-1)$$

Step 3: Simplify and solve for the variable.

$$m(m-1)-3m=4(m+1)(m-1)$$
$$m^2-m-3m=4m^2-4$$
$$3m^2+4m-4=0$$
$$(3m-2)(m+2)=0$$
$$m=\frac{2}{3} \text{ or } m=-2$$

Since the non-permissible values for m are −1 and 1, both solutions $m=\frac{2}{3}$ and $m = -2$ are solutions.

- Some possible activities that reinforce working with rational equations are described below.
  - Provide each student with a rational equation. Ask them to identify the non-permissible roots and then solve the equation. Ask students not to write their name on the paper since student solutions will be collected and redistributed randomly around the room. Students should verify if the solution is correct. If the solution is incorrect, ask students to identify the error and write the correct solution.
  - Create pairs of cards with word problems and matching equations to solve the word problems. Distribute the cards amongst the students and have them find their partner by matching the word problem with the corresponding equation. Once students have found their partner, they should work in their pairs to solve the equation and verify their solution.

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- rational equation

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 6.4, pp. 341–351

Notes

Trigonometry
10 hours

GCO: Students will be expected to develop trigonometric reasoning.

SCO T01 Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°). [R, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

T01.01 Sketch an angle in standard position, given the measure of the angle.
T01.02 Determine the reference angle for an angle in standard position.
T01.03 Explain, using examples, how to determine the angles from 0° to 360° that have the same reference angle as a given angle.
T01.04 Illustrate, using examples, that any angle from 90° to 360° is the reflection in the x-axis and/or the y-axis of its reference angle.
T01.05 Determine the quadrant in which a given angle in standard position terminates.
T01.06 Draw an angle in standard position given any point $P(x, y)$ on the terminal arm of the angle.
T01.07 Illustrate, using examples, that the points $P(x, y)$, $P(-x, y)$, $P(-x, -y)$, and $P(x, -y)$ are points on the terminal sides of angles in standard position that have the same reference angle.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
M04 Students will be expected to develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. (M10)* **G03** Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case. (M11)**.	**T01** Students will be expected to demonstrate an understanding of angles in standard position (0° to 360°).	**T01** Students will be expected to demonstrate an understanding of angles in standard position, expressed in degrees and radians.

* M10—Mathematics 10
** M11—Mathematics 11

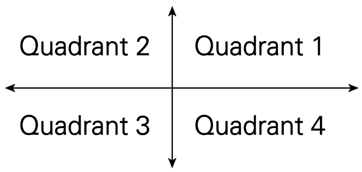
Background

In Mathematics 10, students used the Pythagorean theorem and the primary trigonometric ratios to find missing side lengths and angle measures in right triangles. They also investigated and worked with angles of elevation and depression (M04).

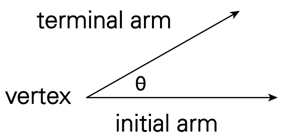
In Mathematics 11, students used the law of sines (including the ambiguous case) and the law of cosines to work with problems involving oblique triangles.

In this unit, students will evaluate the primary trigonometric ratios for angles from 0° to 360° using the coordinate plane.

The x-axis and y-axis divide a plane into four quadrants. The axes are the boundaries of these four quadrants and are not considered to be part of the quadrants.



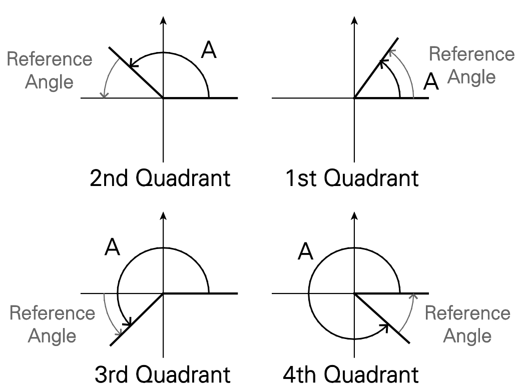
Students will be introduced to terminology such as **rotation angle**, **initial arm**, **terminal arm**, **vertex**, and **standard position**. They will sketch angles in standard position on the coordinate plane and identify the quadrant where the terminal arm lies.



A **rotation angle** is formed by rotating an **initial arm** through an angle (θ) about a fixed point called a **vertex** to a terminal position called the **terminal arm**.

With respect to the coordinate axes, a rotation angle is in **standard position** if the initial arm is on the positive x-axis and the vertex is at the origin. A positive angle results from a counter-clockwise rotation of the terminal arm and a negative angle results from a clockwise rotation of the terminal arm.
(**Note:** Negative rotational angles and co-terminal angles are not part of this outcome. They will be addressed in Pre-calculus 12.)

The **reference angle** is the positive acute angle that can represent an angle of any measure. Any angle on the coordinate plane has a reference angle between 0° and 90°. The reference angle is always the smallest angle that the terminal arm makes with the x-axis. **Quadrantal angles** are those special angles formed by the intersection of the x-axis and y-axis (i.e., 0°, 90°, 180°, 270°, and 360°).

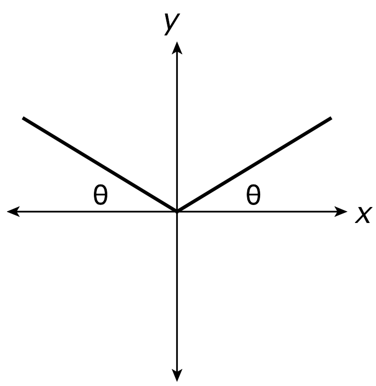
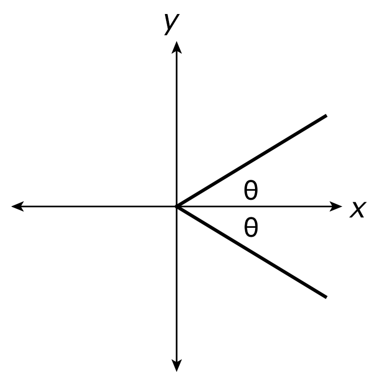
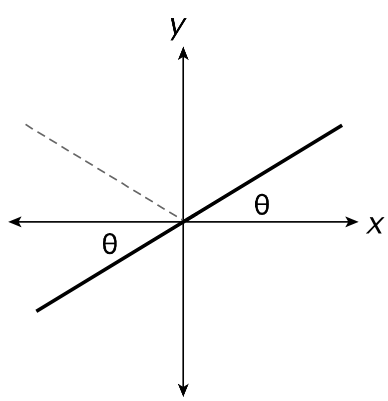


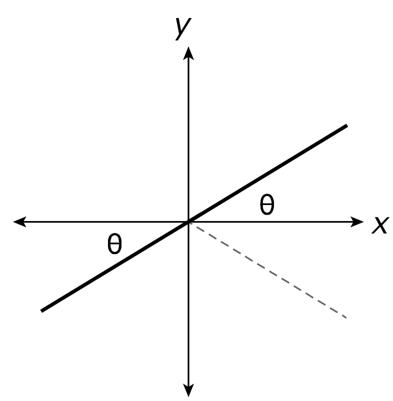
An understanding of **reference angles** will be critical when students compare the trigonometic ratio of an angle in standard position to the trigonometric ratio of its reference angle. It will also be useful when determining the exact trigonometric ratios for angles in standard position that are multiples of 30°, 45°, and 60°, and when solving equations of the form $\sin\theta = a$ or $\cos\theta = a$.

Students should be given ample time to use and become confident in using a coordinate axis, determining reference angles, drawing angles in standard position, and locating the quadrant that contains the terminal arm. They should be able to use the terminology easily and effectively with reference to the coordinate axis.

It is important that students understand how to find the reference angle without memorizing formulas. Having students draw the angle and observe the reference angle using reflections is critical to this process.

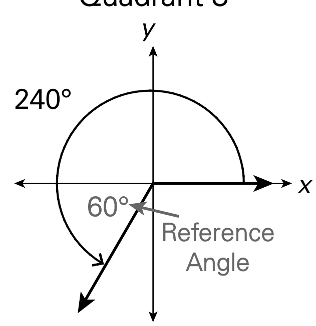
Any angle on a coordinate axis between 90° and 360° is a reflection of a reference angle in the *x*-axis or *y*-axis (or both).

<table>
<tr><td>Reflected in the y-axis
Angle between 90° and 180°</td><td>Reflected in the x-axis
Angle between 270° and 360°</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td colspan="2">Reflected in both the x-axis and the y-axis. Angle between 180° and 270°.</td></tr>
<tr><td colspan="2">

OR

</td></tr>
</table>

Students should be able to illustrate this on a coordinate axis and determine the reference angle. For example the reference angle for 240° is 60°, which is a reflection of the 60° angle in both the *x*- and the *y*-axis and could be determined algebraically by solving the equation 240° = 180° + θ. This concept can be extended to any point (x, y) and its reflection in either the *x*-axis or *y*-axis (or both).

Quadrant 3



Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Sketch an angle of approximately 60° and one of approximately 100°.
- As an angle size increases from zero degrees to ninety degrees, the value of sine increases but the value of cosine decreases. Explain why this happens.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the reference angle for each of the following angles:
  (a) 100°
  (b) 250°
  (c) 315°

- How are the angles formed with the *x*-axis and the segment connecting the origin to each of the points (–3, 4), (–3, –4) and (3, –4) related to each other? Explain your reasoning.

- The terminal arm of an angle in standard position is *R* units long. State the coordinates of the endpoint on this terminal arm when the angle or rotation is 0°, 90°, 180°, 270°, or 360°?

- Sketch a 310° angle.
  (a) Determine which quadrant contains the terminal arm.
  (b) Find its reference angle.
  (c) Find two other angles that have the same reference angle.

- Mark point (–5, –2) on a coordinate grid.
  (a) Name the quadrant that contains this point.
  (b) Draw the terminal arm from the point to the origin and label the reference angle.
  (c) State the coordinates two points, in other quadrants, that have the same reference angle.

- Working with a partner, explore angles and their reference angles through the following task.
  Step 1: Draw an x-axis and y-axis on a sheet of loose leaf using a ruler.
  Step 2: Draw an angle of 135° in standard position.
  Step 3: Using a pencil, shade from the terminal arm to the x-axis.
  Step 4: Fold the loose leaf on the y-axis and press firmly to transfer the lead.
  Step 5: Unfold the paper and measure the angle between the initial arm and the terminal arm to give the reference angle.

  Extension: Repeat this with any angle between 90° and 360° to determine the reference angle. Look for patterns that would help to determine the reference angle without drawing.
  Note: This same task can be repeated for the reflection of any point (x, y).
- Sketch the reference angle that would correspond to (3, –5).

- Explain how you can use reference angles to determine the trig ratios of any angle.

- Explain why there are only two angles between 0° and 360° that have the same cosine ratio.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Teachers create a model of the coordinate grid with an initial arm and terminal arm that can be moved physically. Ask students to demonstrate the placement of the terminal arm when given the measure of an angle between 0° and 360°. They should explore acute, obtuse, right, straight, and reflex angles.

- Provide students with angles sketched in standard position from Quadrants 2, 3, and 4, such as 150°, 210°, and 330°. Ask them to determine the related reference angle and explain why the reference angle for all three is the same.

- Once students have sketched an angle in standard position and determined the quadrant in which it terminates, it is a natural extension to draw an angle given a point on its terminal arm. Students will determine which quadrant the terminal arm of the angle is located based on the point $P(x, y)$ given. Students should be given an opportunity to graph a point and reflect it in the y-axis, in the x-axis, and about in both the x-axis and the y-axis. The various reflections will illustrate how a point on the terminal arm of (1, 2), for example, reflects to become (–1, 2) in quadrant 2, (–1, –2) in quadrant 3 and (1, –2) in quadrant 4.

- To determine the reference angle using reflections in the x- and y-axis, have students accurately draw an angle between 90° and 360° on a coordinate grid. Then have them fold the paper on either the x- or y-axis (or both) until they can determine the reference angle, using a protractor. Students should use a pencil for this task so that the lead will transfer with each fold.

- Students can use this same process (paper folding) to illustrate the reflection of point (x, y) in all quadrants and determine that the reference angle formed as a reflection of the point on a terminal arm is the same.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper
- protractor
- tracing paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- initial arm
- quadrantal angle
- quadrants
- reference angle
- rotation angle
- standard position
- terminal arm
- vertex

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 2.1, pp. 74–87

Notes

SCO T02 Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position. [C, ME, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

T02.01 Determine, using the Pythagorean theorem or the distance formula, the distance from the origin to a point $P(x, y)$ on the terminal arm of an angle.

T02.02 Determine the value of $\sin\theta$, $\cos\theta$, or $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ.

T02.03 Determine, without the use of technology, the value of $\sin\theta$, $\cos\theta$, or $\tan\theta$, given any point $P(x, y)$ on the terminal arm of angle θ, where θ = 0°, 90°, 180°, 270°, or 360°.

T02.04 Determine the sign of a given trigonometric ratio for a given angle, without the use of technology, and explain.

T02.05 Solve, for all values of θ, an equation of the form $\sin\theta = a$ or $\cos\theta = a$, where $-1 \le a \le 1$, and an equation of the form $\tan\theta = a$, where a is a real number.

T02.06 Determine the exact value of the sine, cosine, or tangent of a given angle with a reference angle of 30°, 45°, or 60°.

T02.07 Describe patterns in and among the values of the sine, cosine, and tangent ratios for angles from 0° to 360°.

T02.08 Sketch a diagram to represent a problem.

T02.09 Solve a contextual problem, using trigonometric ratios.

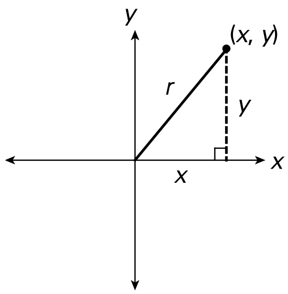
Scope and Sequence

Mathematics 10	Pre-calculus 11	Pre-calculus 12
M04 Students will be expected to develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. **RF08** Students will be expected to solve problems that involve the distance between two points and the midpoint of a line segment.	**T02** Students will be expected to solve problems, using the three primary trigonometric ratios for angles from 0° to 360° in standard position.	**T02** Students will be expected to develop and apply the equation of the unit circle. **T03** Students will be expected to solve problems using the six trigonometric ratios for angles expressed in radians and degrees.

Background

In Mathematics 10 students solved problems using the Pythagorean theorem (G02), the three primary trigonometric ratios (G03), and the distance formula (RF08). This outcome extends the students prior knowledge of trigonometric ratios to points located on the coordinate plane. They will solve problems for angles from 0° to 360°. Angles greater than 360° will be addressed in Pre-calculus 12.

If θ is an angle in standard position, and point P (x, y) is a point on the terminal arm of angle θ, students will use the Pythagorean theorem to determine the length of the hypotenuse r. The three primary trigonometric ratios will be defined in terms of x, y, and r.



$\sin\theta = \dfrac{\text{opposite side}}{\text{hypotenuse}} = \dfrac{y}{r}$ $\quad\cos\theta = \dfrac{\text{adjacent side}}{\text{hypotenuse}} = \dfrac{x}{r}$ $\quad\tan\theta = \dfrac{\text{opposite side}}{\text{adjacent side}} = \dfrac{y}{x}$

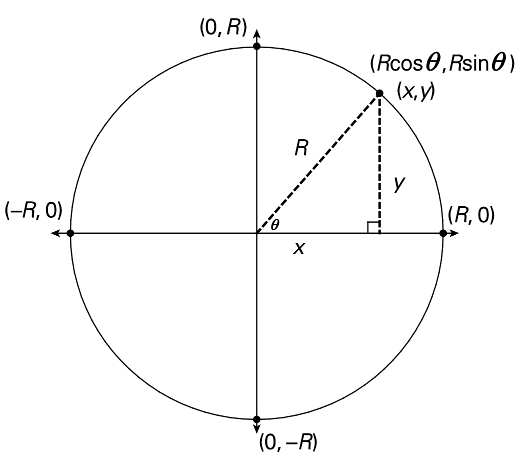
Special angles formed by the intersection of the x-axis and y-axis will also be investigated (i.e., quadrantal angles). Trigonometric ratios for angles whose measurements are 0°, 90°, 180°, 270°, or 360° will be explored.

Students will understand that any point (x, y) can be represented by $(r\cos\theta,\ r\sin\theta)$ where r represents the distance the point (x, y) is from $(0, 0)$.

$\cos\theta = \dfrac{x}{r} \qquad \sin\theta = \dfrac{y}{r}$

$r\cos\theta = x \qquad r\sin\theta = y$

Using a circle with radius R would be an effective strategy to use when discussing quadrantal angles. It should be noted that the unit circle is not formally introduced until Pre-calculus 12.



Encourage students to use the diagram to help them write the values for sine, cosine, and tangent for each quadrantal angle. For example,

$\cos 30^\circ$, $\sin 180^\circ = \dfrac{0}{R} = 0$, and $\tan 180^\circ = \dfrac{0}{-R} = 0$

The students should also understand that $\tan\theta = \frac{y}{x} = \frac{\text{rise}}{\text{run}}$ and therefore $\tan\theta$ represents the slope of the line passing through the points (*x*, *y*) and (0, 0).

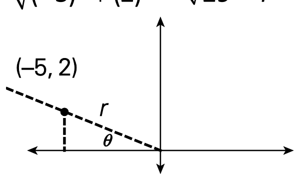
Thus the slope of the line passing through (0, 0), (1, 1), (2, 2), and (3, 3) is 1; therefore, we can observe that $\tan 45^\circ = 1$.

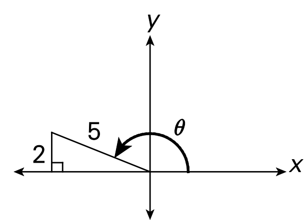
It is important that students understand that the length of the terminal arm does not affect the values of the cosine, sine, and tangent ratios. This can be done by asking students to determine the trigonometric ratios for points such as the following:

- (1, 0), (2, 0) , and (3, 0)
- (0, 1), (0, 2), and (0, 3)
- (–1, 0), (–2, 0), and (–3, 0)
- (0, –1), (0, –2), and (0, –3)
- (1, 2), (2, 4), (3, 6), and (2, 8)

Students should recognize that, except for quadrantal angles, every angle drawn in standard position has a corresponding reference angle.

Given either a point on the terminal arm of an angle or the value of one trigonometric ratio and the quadrant, students will be able to determine the exact value of another trigonometric ratio without determining the size of the angle.

<table>
<tr><th>Given a point on the terminal arm</th><th>Given one trigonometric ratio and the quadrant</th></tr>
<tr><td>(x, y) = (–5, 2)
$\sqrt{(-5)^2 + (2)^2} = \sqrt{29} = r$


$\sin\theta = \frac{2}{\sqrt{29}}$ $\cos\theta = \frac{-5}{\sqrt{29}}$ $\tan\theta = \frac{2}{-5}$</td><td>$\sin\theta = \frac{2}{5}$; quadrant 2
$\sin\theta = \frac{2}{5}$; $y = 2$, $r = 5$


$5^2 = 2^2 + x^2$
$25 - 4 = x^2$ (x is negative in quadrant 2)
$x = \pm\sqrt{21}$
$\therefore \cos\theta = \frac{-\sqrt{21}}{5}$ and $\tan\theta = \frac{2}{-\sqrt{21}}$</td></tr>
</table>

In addition to solving for trigonometric ratios, students will be expected to solve simple trigonometric equations of the forms $\sin\theta = a$, $\cos\theta = a$, and $\tan\theta = a$ to determine the value of θ. Students will also be exposed to problems where they will have to rearrange the equation before they solve it.

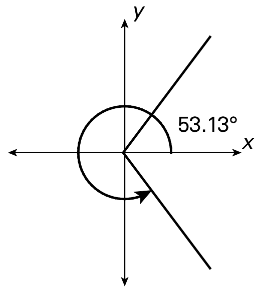
For example students will be expected to solve, using technology, the trigonometric equation $5\cos\theta - 1 = 2$.

$$5\cos\theta - 1 = 2$$
$$5\cos\theta = 3$$
$$\cos\theta = \frac{3}{5}$$

Since $\cos\theta > 0$, the angle is in quadrant 1 or quadrant 4.

Using technology, students will obtain the angle 53.13° (the solution in quadrant 1).

Sketching a graph will enable students to recognize the quadrant 4 solution as being 360° – 53.13° = 306.87°.



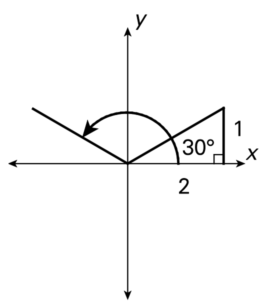
Reference angles of 30°, 45°, or 60° occur frequently in problems. Using an isosceles right and an equilateral triangle students will determine the exact values of the trigonometric value for each of the **special angles**, 30°, 45°, and 60°. They would then be expected to solve equations, such as $2\sin\theta + 4 = 5$ without the use of technology.

$2\sin\theta + 4 = 5$
$2\sin\theta = 1$
$\sin\theta = \dfrac{1}{2}$

Since $\sin\theta > 0$, the angle is in quadrant 1 or quadrant 2.

Students will know that $\sin 30^\circ = \dfrac{1}{2}$ (the solution in quadrant 1).

Sketching a graph will enable students to recognize the quadrant 2 solution as being 180° – 30° = 150°.



Students will develop patterns to determine the exact value of the sine, cosine, or tangent for angles sharing the same reference angle.

For example,

- 60° is the reference angle for 120°, 240°, and 300°. Since $\sin 60^\circ = \dfrac{\sqrt{3}}{2}$, the sine of these angles will have the same absolute value but be negative or positive depending upon the quadrant in which it sits.

  $\sin 120^\circ = \dfrac{\sqrt{3}}{2}$ $\quad\quad$ $\sin 240^\circ = -\dfrac{\sqrt{3}}{2}$ $\quad\quad$ $\sin 300^\circ = -\dfrac{\sqrt{3}}{2}$

Students are also expected to be able to solve simple trigonometric equations resulting in a quandrantal angle of 0°, 90°, 180°, 270°, and 360°, without the use of technology.

For example 2 tan θ + 1 = 1.

2 tan θ + 1 = 1
2 tan θ = 0
tan θ = 0
θ = 0° or 180°

Notes:

- Avoid the use of memorization shortcuts such as CAST. Instead encourage students to think about the sign of *x* and *y* in each of the quadrants. For example, students should understand that cos120°will be negative since the *x*-coordinate is negative in the second quadrant.

- Avoid memorization of a chart such as the following:

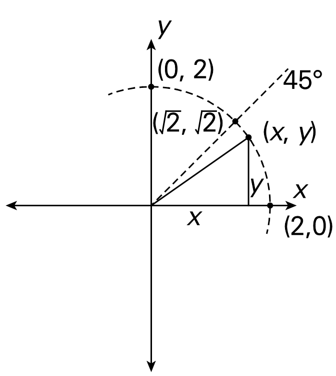
θ	30°	45°	60°
sin θ	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$
cos θ	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$	$\frac{1}{2}$
tan θ	$\frac{1}{\sqrt{3}}=\frac{\sqrt{3}}{3}$	1	$\frac{\sqrt{3}}{1}$

It is important that students be able to work out these values using an equilateral triangle and isosceles right triangle or by drawing a picture and using what they know about the relationship between the point on the terminal arm (*x*, *y*) and the trigonometric ratios as shown below.

That is, $\cos\theta=\frac{x}{r}$, $\sin\theta=\frac{y}{r}$, and $\tan\theta=\frac{y}{x}$.

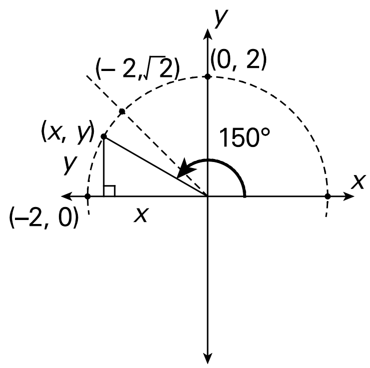
When *r* = 2, $\cos\theta=\frac{x}{2}$, $\sin\theta=\frac{y}{2}$, and $\tan\theta=\frac{y}{x}$.

For example, to determine the exact value of $\sin 30^\circ$, sketch a picture.



From the sketch, it is evident that *y* is smaller than *x*. Therefore, since the exact values that we have to choose from are $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$ or $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$, and since we can conclude that for a situation with *r* = 2 the *y*-coordinate is larger than 0 and less than $\sqrt{2}$, we know that $\sin 30^\circ=\frac{1}{2}$.

For example, to determine the exact value of $\cos 150^\circ$, sketch a picture.



From the sketch, it is evident that *y* is smaller than *x*. Therefore, since the exact values that we have to choose from are $0, \frac{1}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, 1$ or $\frac{0}{2}, \frac{\sqrt{1}}{2}, \frac{\sqrt{2}}{2}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{2}$, and since we can conclude that for a situation with $r = 2$ the *x*-coordinate's absolute value is larger than $\sqrt{2}$ and less than 2, we know that $\cos 150^\circ = -\frac{\sqrt{3}}{2}$.

Students will graph the functions $y = \sin\theta$, $y = \cos\theta$, and $y = \tan\theta$ in Pre-calculus 12 (T04). In this course they should be given an opportunity to explore the patterns in the sine, cosine, and tangent ratios.

For example, is expected that they will notice
- that as the angle rotates from 0° to 90°, the sine ratio increases from 0 to 1, the cosine ratio decreases from 1 to 0, and the tangent ratio increases from 0 to infinite.
- that as the angle rotates from 90° to 180°, the sine ratio decreases from 1 to 0, the cosine ratio begins at 0 and moves to –1, and the tangent ratio is negative and changes from infinite to 0.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Find the distance between (0, 0) and (1, 2).
- Determine the height of an equilateral triangle with a side length of 10 cm.
- What are the measures of the angles in an equilateral triangle?
- What are the measures of the angles in an isosceles right triangle?

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- If θ is an angle in standard position, summarize in which quadrant(s) the terminal arm of θ will lie if
  - $\tan\theta > 0$ or $\tan\theta < 0$
  - $\sin\theta > 0$ or $\sin\theta < 0$
  - $\cos\theta > 0$ or $\cos\theta < 0$

- Explain how you can determine the sign (±) of the various ratios in each of the quadrants. Share and compare your findings with the rest of the class.

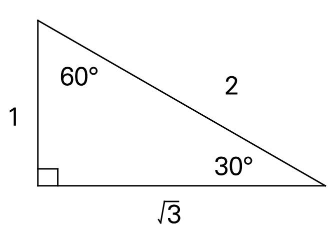
- Solve each of the following equations, with the use of technology, for $0° \le \theta \le 360°$.
  (a) $2\tan\theta = 3$
  (b) $3\sin\theta + 2 = 1$
  (c) $2 - 5\cos\theta = 1$
  (d) $\sin\theta = -\frac{4}{5}$

- Solve each of the following equations, without the use of technology, for $0° \le \theta \le 360°$.
  (a) $\tan\theta = \sqrt{3}$
  (b) $2\sin\theta = 1$
  (c) $2 - \cos\theta = 1$
  (d) $\sin\theta = -\frac{\sqrt{3}}{2}$
  (e) $2(\cos\theta)^2 - 1 = 0$
  (f) $2\sin\theta + 3 = 5$
  (g) $2\tan\theta + 3 = 3$

- Explain how solving the linear equation $-2x - 1 = 0$ is similar and how it is different from solving the trigonometric equation $-2\sin\theta - 1 = 0$.

- When evaluating $\sin 45°$, explain why $\frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ is called an exact value while 0.707 is approximate.

- Working with the 30° - 60° - 90° triangle shown below, answer the following questions.



  (a) Calculate the exact value of $\sin 30°$ and $\cos 30°$
  (b) Calculate the exact value of $\sin 60°$ and $\cos 60°$.

(c) Determine $\sin 30^\circ$, $\cos 30^\circ$, $\sin 60^\circ$, and $\cos 60^\circ$ using a calculator. What do you notice?
(d) Which is greater, $\sin 60^\circ$ or $\sin 30^\circ$? Why?
(e) What is the same and what is different about $\sin 30^\circ$ and $\sin 150^\circ$?
(f) What is the same and what is different about $\cos 30^\circ$ and $\cos 150^\circ$?
(g) What do you notice when you compare the exact value of $\tan 30^\circ$ and $\tan 60^\circ$ using the right triangle?
(h) What is the same and what is different about $\tan 30^\circ$ and $\tan 330^\circ$?

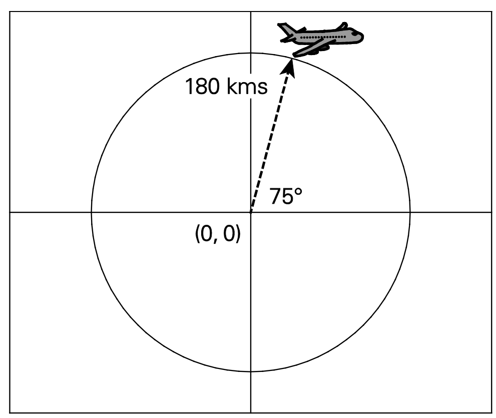
- Explain how you can determine if $\sin\theta$ is positive or negative without using a calculator.

- Explain how you can determine if $\cos\theta$ is positive or negative without using a calculator.

- Explain how you can determine if $\tan\theta$ is positive or negative without using a calculator.

- Without using a calculator, determine each of the following:
  (a) If $\sin\theta = \frac{1}{3}$; find $\cos\theta$.
  (b) If $\tan\theta = \frac{4}{3}$; find $\sin\theta$.
  (c) If $\cos\theta = -\frac{2}{3}$; find $\sin\theta$.
  (d) If $\sin\theta = -\frac{\sqrt{2}}{3}$; find $\tan\theta$.
  (e) For point (5, 12), find the exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$.
  (f) For point (–3, 7), find the exact values for $\sin\theta$, $\cos\theta$, and $\tan\theta$.

- Radar can locate aircraft hundreds of km away. A radar screen is divided into pixels that can be translated into *x*-coordinate and *y*-coordinate (1 pixel represents 1 kilometre). If an aircraft is at an angle of 75° with the horizontal and a distance of 180 kilometres from the tower located at (0, 0), which pixel (*x*-coordinate and *y*-coordinate) on the monitor would represent the position of the aircraft relative to the tower?



FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should be used to inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

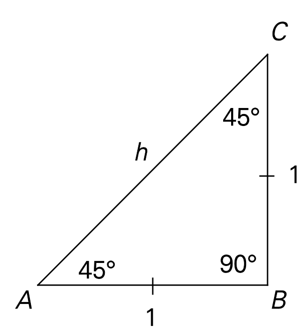
SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- A task such as the following provides an opportunity for students to explore the trigonometric ratios given any point on the terminal arm of angle θ. Begin with the point (–4, –5). Continue the task to ensure that endpoints from all quadrants are used. Use the following directions and questions to guide students.
  - Plot the given point *P* (–4, –5). Which quadrant does this point lie in?
  - Construct the corresponding angle in standard position.
  - Drop a perpendicular to the *x*-axis creating a right triangle. Which value represents the adjacent side? Which value represents the opposite side?
  - How can you determine the exact length of the hypotenuse?
  - State the cosine, sine, and tangent ratios associated with the angle.
  - What determines the sign of the ratio? Explain your reasoning.

- Use the following questions to guide students through the solution of an equation such as $\cos\theta = \frac{1}{4}$.
  - Determine which quadrants contain solutions. Are there any restrictions?
  - Use technology to determine the reference angle for the given value of $\cos\theta = \frac{1}{4}$.
  - Determine the measure of the related angles in standard position where 0° ≤ θ ≤ 360°.

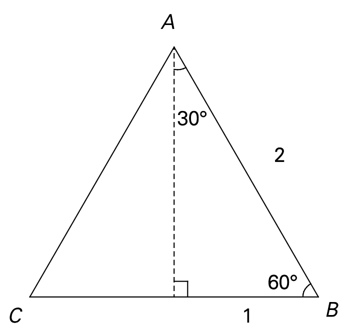
- Geometric properties of right triangles containing 30°, 45°, and 60° can be used to obtain the trigonometric values. To demonstrate the exact trigonometric value for θ = 45°, students can construct an isosceles right triangle with the smallest side measuring 1 unit. Students can then use the Pythagorean theorem to determine the length of the hypotenuse, 2.



Consider the following guiding questions:
- What is the exact value of $\sin 45^\circ$?
- What do you notice if you evaluate $\sin 45^\circ$ with a calculator?
- What do you notice about the values of $\sin 45^\circ$ and $\cos 45^\circ$? Can you explain why they have the same value?
- What is the value of $\tan 45^\circ$? Can you determine the value of $\tan 90^\circ$ using the right triangle? Explain your reasoning.
- What is the same and what is different about $\sin 45^\circ$ and $\sin 225^\circ$?
- What is the same and what is different about $\cos 45^\circ$ and $\cos 135^\circ$?
- What is the same and what is different about $\tan 45^\circ$ and $\tan 135^\circ$?

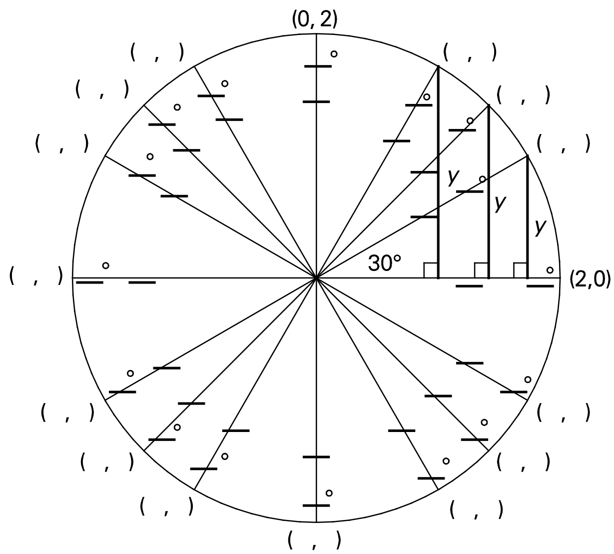
When determining the exact value of the trigonometric ratios of a given angle, students will sometimes result in a value where the denominator contains a radical. Consider the example, $\cos 45^\circ = \frac{1}{\sqrt{2}}$. Depending on the order in which this course is taught, students may not have experience rationalizing the denominator.

- A similar strategy to demonstrate the exact trigonometric values for θ = 30° or θ = 60° is for students to construct an equilateral triangle with a side length of 2. Drawing an altitude from vertex *A*, students can then use the Pythagorean theorem to determine the length of the altitude, 3.

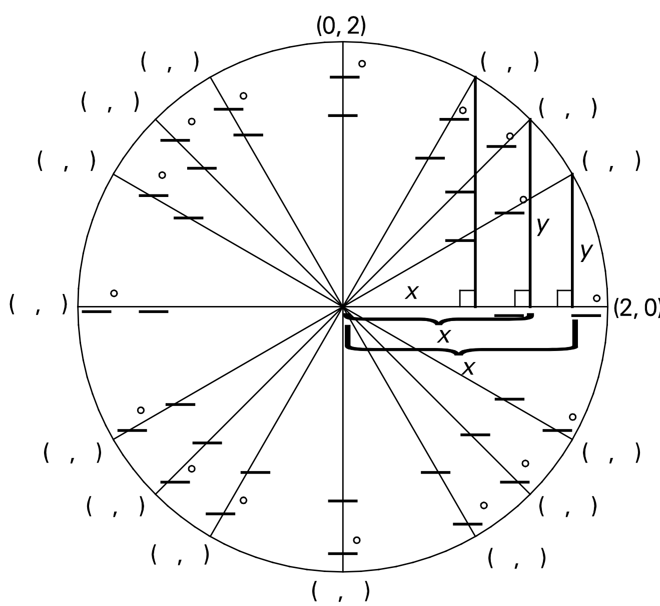


Consider the following guiding questions:
- What is the exact value of $\sin 30^\circ$, $\sin 60^\circ$, $\cos 30^\circ$, $\cos 60^\circ$, $\tan 30^\circ$, and $\tan 60^\circ$?
- What do you notice if you evaluate $\sin 30^\circ$, $\sin 60^\circ$, $\cos 30^\circ$, $\cos 60^\circ$, $\tan 30^\circ$, and $\tan 60^\circ$ with a calculator?
- What do you notice about the values of sin30° and $\cos 60^\circ$? Can you explain why they have the same value?
- What is the same and what is different about $\sin 30^\circ$, $\sin 150^\circ$, $\sin 210^\circ$, and $\sin 330^\circ$?
- What is the same and what is different about $\cos 30^\circ$, $\cos 150^\circ$, $\cos 210^\circ$, and $\cos 330^\circ$?
- What is the same and what is different about $\tan 30^\circ$, $\tan 150^\circ$, $\tan 210^\circ$, and $\tan 330^\circ$?

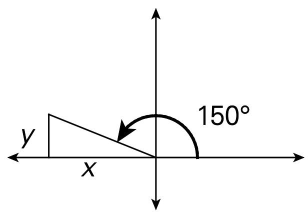
- It is helpful to use a circle with a radius of 2 to draw students' attention to how the *y*-values progress from 0 to 2 as the angle increases from 0° to 90°. Thus, the $\sin\theta$ increases from $\frac{0}{2}$ or 0 to $\frac{2}{2}$ or 1 as the angle increases from 0° to 90°.



- Similarly, it is helpful to use a circle with a radius of 2 to draw students' attention to how the *x*-values progress from 2 to 0 as the angle increases from 0° to 90°. Thus the $\cos\theta$ decreases from $\frac{2}{2}$ or 1 to $\frac{0}{2}$ or 0 as the angle increases from 0° to 90°.



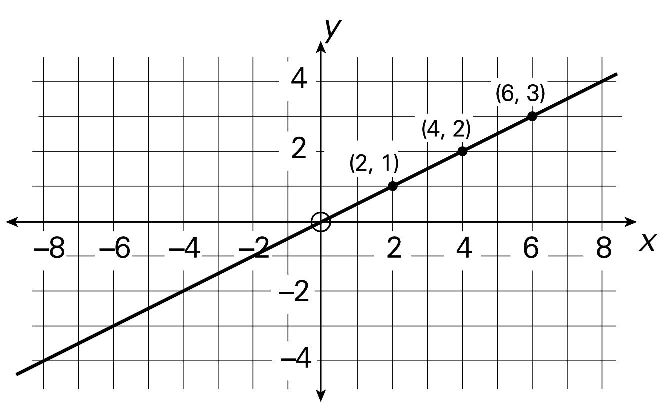
- Encourage students to use mental mathematics to determine the trigonometric function values whenever the terminal side makes a 30°, 45°, and 60° angle with the *x*-axis. In other words, since students know the special angles are associated with the sine and cosine ratios of 0, $\pm\frac{1}{2}$, $\pm\frac{\sqrt{2}}{2}$, $\pm\frac{\sqrt{3}}{2}$, ±1 or $\pm\frac{\sqrt{0}}{2}$, $\pm\frac{\sqrt{1}}{2}$, $\pm\frac{\sqrt{2}}{2}$, $\pm\frac{\sqrt{3}}{2}$, $\pm\frac{\sqrt{4}}{2}$, the specific values can be determined by looking at a sketch of the angle of interest. For example, cos150° would be drawn as follows:



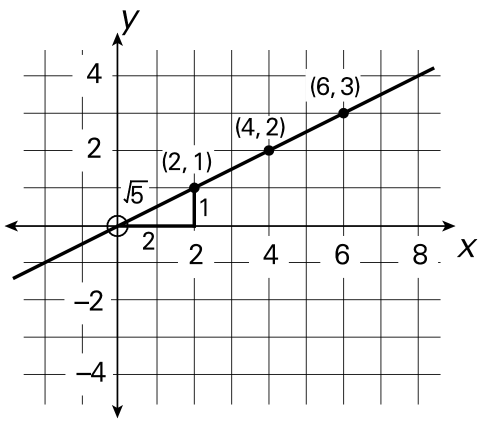
Since the *x* is larger than the *y*, $\cos 150^\circ$ must be larger than $\sin 150^\circ$, so it must be $\pm\frac{\sqrt{3}}{2}$. It is evident from the sketch that cosine is negative in the second quadrant, therefore $\cos 150^\circ = -\frac{\sqrt{3}}{2}$.

- To encourage students to solve problems using the trigonometric ratios, ask them to draw a sketch of a diagram to help them gain a visual understanding of the problem. Consider an example such as the following:
  - The arm of a crane used for lifting very heavy objects can move so that it has a minimum angle of inclination of 30° and a maximum of 60°. Use exact values to find an expression for the change in the vertical displacement of the end of the arm, in terms of the length of the arm, *a*.

- To develop $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$, and $\tan\theta = \frac{y}{x}$, have students draw a line that passes through the origin such as the one shown below.



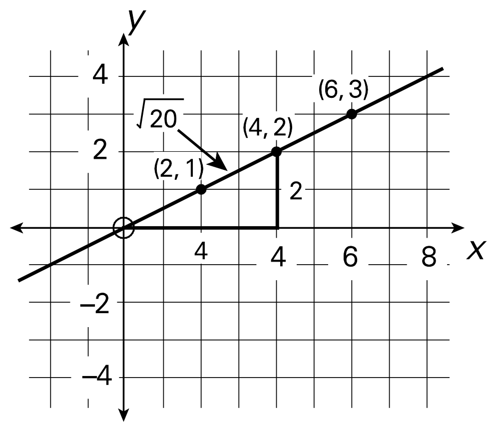
Ask students to drop a perpendicular from the *y*-coordinate of each of the points on this line to the *x*-axis and determine the hypotenuse (*r*) of each of the triangles formed. Then have them use the triangle created to determine the sine, cosine, and tangent of the angle θ as illustrated below.



$$\cos\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\sin\theta = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

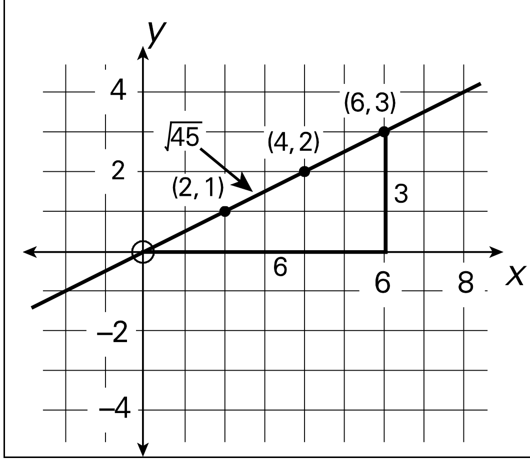
$$\tan\theta = \frac{1}{2}$$



$$\cos\theta = \frac{4}{\sqrt{20}} = \frac{4}{2\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\sin\theta = \frac{2}{\sqrt{20}} = \frac{2}{2\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{2}{4} = \frac{1}{2}$$



$$\cos\theta = \frac{6}{\sqrt{45}} = \frac{6}{3\sqrt{5}} = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$$

$$\sin\theta = \frac{3}{\sqrt{45}} = \frac{3}{3\sqrt{5}} = \frac{1}{\sqrt{5}} = \frac{\sqrt{5}}{5}$$

$$\tan\theta = \frac{3}{6} = \frac{1}{2}$$

Students should quickly recognize that the $\sin\theta$, $\cos\theta$, and $\tan\theta$ will be the same regardless of where on the terminal arm the point is located.

- Divide the class into two groups. Individual students in one group will be given a card containing trigonometric ratios of special angles (e.g., $\sin 60°$, $\cos 120°$). In the other group, students should be given associated exact values (e.g., $\frac{\sqrt{3}}{2}$, $-\frac{\sqrt{3}}{2}$). Ask students to find a partner to form a matching pair.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- exact values
- quadrantal angles

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 2.1 and 2.2, pp. 74–99

Notes

SCO T03 Students will be expected to solve problems, using the cosine law and sine law, including the ambiguous case. [C, CN, PS, R, T]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

T03.01 Sketch a diagram to represent a problem that involves a triangle without a right angle.
T03.02 Solve, using primary trigonometric ratios, a triangle that is not a right triangle.
T03.03 Explain the steps in a given proof of the sine law or cosine law.
T03.04 Sketch a diagram and solve a problem, using the cosine law.
T03.05 Sketch a diagram and solve a problem, using the sine law.
T03.06 Describe and explain situations in which a problem may have no solution, one solution, or two solutions.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
M04 Students will be expected to develop and apply the primary trigonometric ratios (sine, cosine, tangent) to solve problems that involve right triangles. (M10)* **G03** Students will be expected to solve problems that involve the cosine law and the sine law, including the ambiguous case. (M11)**	**T03** Students will be expected to solve problems, using the cosine law and sine law, including the ambiguous case. (This outcome has already been covered by students in Mathematics 11. It can be omitted for this course.)	**T06** Students will be expected to prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent)

* M10—Mathematics 10
** M11—Mathematics 11

Background

This outcome has been addressed in Mathematics 11.

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 2.3 and 2.4, pp. 100–125

Notes

Relations and Functions
60–70 hours

GCO: Students will be expected to develop algebraic and graphical reasoning through the study of relations.

<table>
<tr><td colspan="4">SCO RF01 Students will be expected to factor polynomial expressions of the following form where a, b, and c are rational numbers.
▪ $ax^2 + bx + c,\ a \neq 0$
▪ $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$
▪ $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$
▪ $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$
$b \neq 0$ [CN, ME, R]</td></tr>
<tr><td>[C] Communication
[T] Technology</td><td>[PS] Problem Solving
[V] Visualization</td><td>[CN] Connections
[R] Reasoning</td><td>[ME] Mental Mathematics and Estimation</td></tr>
</table>

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF01.01 Factor a given polynomial expression that requires the identification of common factors.

RF01.02 Determine whether a given binomial is a factor for a given polynomial expression, and explain why or why not.

RF01.03 Factor a given polynomial expression of the form
- $ax^2 + bx + c,\ a \neq 0$
- $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$

RF01.04 Factor a given polynomial expression that has a quadratic pattern, including
- $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$
- $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
AN05 Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. (M10)* **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry (M11)**	**RF01** Students will be expected to factor polynomial expressions of the following form where *a*, *b*, and *c* are rational numbers. ▪ $ax^2 + bx + c,\ a \neq 0$ ▪ $a^2x^2 - b^2y^2,\ a \neq 0,\ b \neq 0$ ▪ $a[f(x)]^2 + b[f(x)] + c,\ a \neq 0$ ▪ $a^2[f(x)]^2 - b^2[g(y)]^2,\ a \neq 0,\ b \neq 0$	**T05** Students will be expected to solve algebraically and graphically, first- and second-degree trigonometric equations with the domain expressed in degrees and radians. **T06** Students will be expected to prove trigonometric identities, using ▪ reciprocal identities ▪ quotient identities ▪ Pythagorean identities ▪ sum or difference identities (restricted to sine, cosine, and tangent) ▪ double-angle identities (restricted to sine, cosine, and tangent)

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
		(continued) **RF11** Students will be expected to demonstrate an understanding of factoring polynomials of degree greater than 2 (limited to polynomials of degree ≤ 5 with integral coefficients).

* M10—Mathematics 10

** M11—Mathematics 11

Background

Students were first exposed to factoring polynomials in Mathematics 10 (AN05). They removed the greatest common factor from the terms of a polynomial and factored polynomials of the form $x^2 + bx + c$ and $ax^2 + bx + c$. Students also factored perfect square trinomials, difference of squares, and trinomials in two variables.

For this outcome, students will extend their knowledge of factoring trinomials and differences of squares to factoring polynomials of the form $a^2x^2 - b^2y^2$, $a \neq 0$, $b \neq 0$, $a[f(x)]^2 + b[f(x)] + c$, $a \neq 0$ and $a^2[f(x)]^2 - b^2[g(y)]^2$, $a \neq 0$, $b \neq 0$.

In Mathematics 10, students worked extensively with factoring polynomial expressions. They were introduced to factoring using concrete and pictorial models and then moved to a symbolic representation. Most of the previous work dealt with integer coefficients. Students should be reminded of the strategies used when factoring polynomial expressions, including the removal of the greatest common factor and the method of decomposition (10AN05).

Students should recognize when factoring expressions such as $x^2 + 6x + 8$, possible binomial factors need to contain factors of 8. Therefore, a student should understand that $(x + 5)$, for example, would not be a possible factor of $x^2 + 6x + 8$. This thinking will improve student factoring skills that will be useful later when using factoring to solve a given quadratic equation, simplify a rational expression, and determine the domain for an irrational expression.

In this course, students will be expected to be proficient with factoring at a symbolic level, including expressions with rational coefficients. Students should also be given the opportunity to apply their own personal strategies. It is necessary that students be able to factor without the use of technology.

When factoring trinomials and differences of squares of the form $a[f(x)]^2 + b[f(x)] + c$, $a \neq 0$ and $a^2[f(x)]^2 - b^2[g(y)]^2$, $a \neq 0$, $b \neq 0$, students can treat the expression as a single variable, continue to factor using their previous skills, and then use substitution to complete the factoring. Alternatively, they can expand and group like terms and then factor.

For example, factor $2(x-2)^2 + 7(x-2) + 5$:

Substitute $p = x - 2$	Expand $(x-2)^2$
$= 2p^2 + 7p + 5$	$= 2(x-2)(x-2)+7(x-2)+5$
$= (2p + 5)(p + 1)$	$= 2(x^2 - 4x + 4) + 7(x-2) + 5$
$= [2(x-2) + 5][(x-2) + 1]$	$= 2x^2 - 8x + 8 + 7x - 14 + 5$
$= (2x - 4 + 5)(x - 2 + 1)$	$= 2x^2 - 1x - 1$
$= (2x + 1)(x - 1)$	$= (2x + 1)(x - 1)$

Students will use the **Factor Theorem** to test whether or not a given binomial is a factor of the trinomial in question. The variable in the binomial factor is solved for and the solution is substituted into the trinomial expression. If this equals zero, then the solution is the root of the expression and the binomial factor is a factor of the trinomial.

This is shown symbolically as follows:
$(x + m)$ is a factor of $ax^2 + bx + c$ **iff** $f(-m) = 0$, where $f(x) = ax^2 + bx + c$.

(**Note: iff** represents "if and only if".)

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

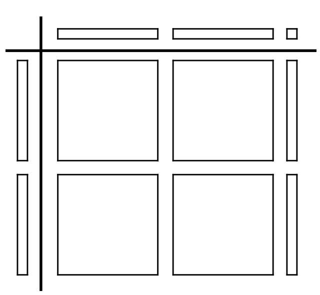
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What is the greatest common factor of 20, 30, and 45?

- Susan drew the picture shown below. What expression was she modelling?



- Factor the following:
  (a) $x^2 - 3x - 4$
  (b) $3x^2 - 14x - 5$
  (c) $9x^2 - 16$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine two values of n that will allow the polynomial $25b^2 - nb + 49$ to be a perfect square trinomial. Use these values to factor the trinomial.
- List possible binomial factors of the expression $x^2 + bx + 24$.
- Explain why $9x^2 - 16y^2$ can be factored but $9x^2 + 16y^2$ cannot be factored.
- Explain why $(2x - 3y)^2 \neq 4x^2 - 9y^2$.
- If $3x^2 - x - k$ is the product of two linear factors with integral coefficients, find the value of k.
- If $2xy + 4y - kx - 10$ is the product of two linear factors with integral coefficients, find the value of k.
- Factor the quadratic $x^2 + (a - \frac{1}{a})x - 1$.
- Given that $(x + m)(3x + n) = 3x^2 + 5xy - 2y^2$, determine the value of $m + n$.
- If $(2x + 1)$ and $(x - 3)$ are factors of $ax^2 + bx + c$, find the values of a and b.
- If $kx^2 - kx - 6$ factors to $k(x + 1)(x + m)$, determine the values of k and m.
- When Jenny factored $x^2 - x - 12$, she said one of her binomial factors was $(x + 5)$. Without actually factoring the trinomial, ask students to explain why her response is incorrect.
- Create an example of a quadratic function that cannot be factored. Explain their reasoning.
- Which type of polynomial did you find easiest to factor and why? Which ones were most difficult and why?
- Factor $4(x + 2)^2 - 9(y + 1)^2$ using two methods. Explain which method you prefer and why.

The following questions illustrate the various types of factoring questions that should be addressed in this unit.

- Fully factor the following:
  (a) $x^2 + 12x + 20$
  (b) $x^2 + 6x - 16$
  (c) $x^2 - 10xy + 24y^2$
  (d) $8x^2 + 33x + 4$
  (e) $6x^2 - 47xy + 15y^2$
  (f) $x^2 - 16$
  (g) $16x^2 - 25y^2$
  (h) $2x^2 - 18y^2$
  (i) $\frac{1}{25}x^3 - \frac{9}{4}x$
  (j) $x^4 + 7x^2 + 12$
  (k) $\sin^2\theta + 2\sin\theta - 3$
  (l) $(2x - 5)^2 + 9(2x - 5) + 18$
  (m) $2(x + 3)^6 + 11(x + 3)^3 + 5$
  (n) $9y^{16} - x^4$
  (o) $144(3x + 5)^4 - 9(2y - 8)^{12}$
  (p) $(x^2 + 2x)^2 - 11(x^2 + 2x) + 24$
  (q) $3^{2x} + 5(3^x) + 6$
  (r) $-3r^2 - r + 10$
  (s) $4x^3 - 64x$

- If $(3x - 1)(x + 2) = 0$, find the values of $(3x - 1)$.

- Factor fully $(x^2 + 3x + 2)(x^2 + 7x + 12) + (x^2 + 4x + 3)$.

- Use the factor theorem for each of the following.
  (a) Is $(x + 2)$ a factor of $x^2 + 7x + 10$?
  (b) Is $(x - 1)$ a factor of $3x^2 + 2x - 5$?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- There are numerous strategies used for factoring, depending upon preference and the structure or the expression. Students should be given ample time to explore the wide range of expression types and the tool kit of strategies they will require. When expressions are quite complex, a substitution often will make it easier for students to see how to proceed. For example, while $(4p^2 – 9)^2 – 5(4p^2 – 9) + 6$ looks quite difficult, replacing $(4p^2 – 9)$ by a different variable, such as k, simplifies the quadratic to $k^2 – 5k + 6$.

- Consider the following example to compare the two methods:

  Factor: $3(x – 1)^2 – 14(x – 1) – 5$

Substitute $p = x – 1$.	Expand $(x – 1)^2$.
$3(x – 1)^2 – 14(x – 1) – 5$ $3p^2 – 14p – 5$ $3p^2 – 15p + 1p – 5$ $3p(p – 5) + 1(p – 5)$ $(p – 5)(3p + 1)$ $[(x – 1) – 5][3(x – 1) + 1]$ $(x – 6)(3x – 2)$	$3(x – 1)^2 – 14(x – 1) – 5$ $3(x^2 – 2x + 1) – 14(x – 1) – 5$ $3x^2 – 6x + 3 – 14x + 14 – 5$ $3x^2 – 20x + 12$ $3x^2 – 18x – 2x + 12$ $3x(x – 6) – 2(x – 6)$ $(x – 6)(3x – 2)$

- Regardless of the method used, remind students that it is important to use parentheses for all substitution.

- When introducing the factor theorem, it is suggested that you have students complete a task such as the following.

  (a) Complete the following charts.

Chart 4

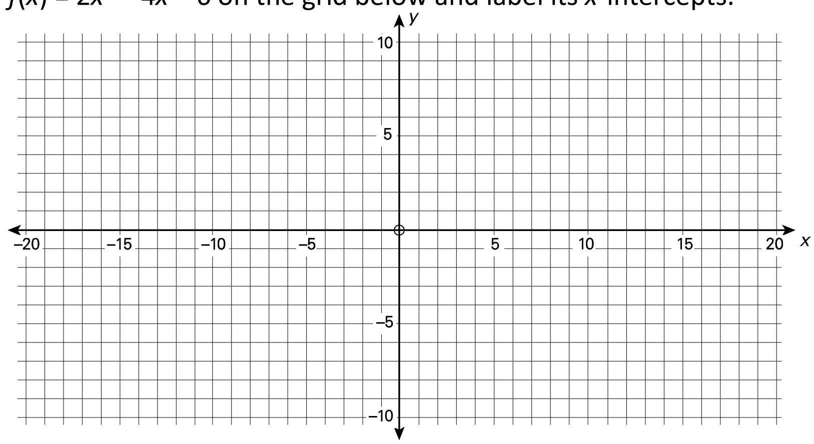
<table>
<tr><td></td><td>$f(-3)$</td><td>$f(-2)$</td><td>$f(-1)$</td><td>$f(0)$</td><td>$f(1)$</td><td>$f(2)$</td><td>$f(3)$</td><td>$f(4)$</td></tr>
<tr><td>$f(x) = 2x^2 - 4x - 6$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td>Factored form of quadratic:</td><td colspan="8" rowspan="2">On the grid below, graph the linear equations that represent the factors of the quadratic. Use technology to sketch $f(x) = 2x^2 - 4x - 6$ on the grid below and label its x-intercepts.

</td></tr>
<tr><td>State each of the linear factors of the quadratic as an equation in the form of $y = mx + b$.</td></tr>
</table>

Chart 5

<table>
<tr><td></td><td>$m(-3)$</td><td>$m(-2)$</td><td>$m(-1)$</td><td>$m(0)$</td><td>$m(1)$</td><td>$m(2)$</td><td>$m(3)$</td><td>$m(4)$</td></tr>
<tr><td>$m(x) = x^2 - 4x + 3$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
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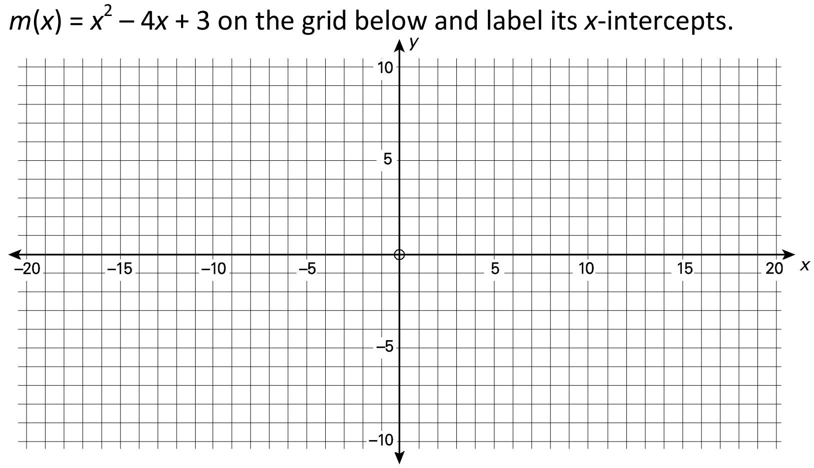

</td></tr>
<tr><td>State each of the linear factors of the quadratic as an equation in the form of $y = mx + b$.</td></tr>
</table>

Chart 6

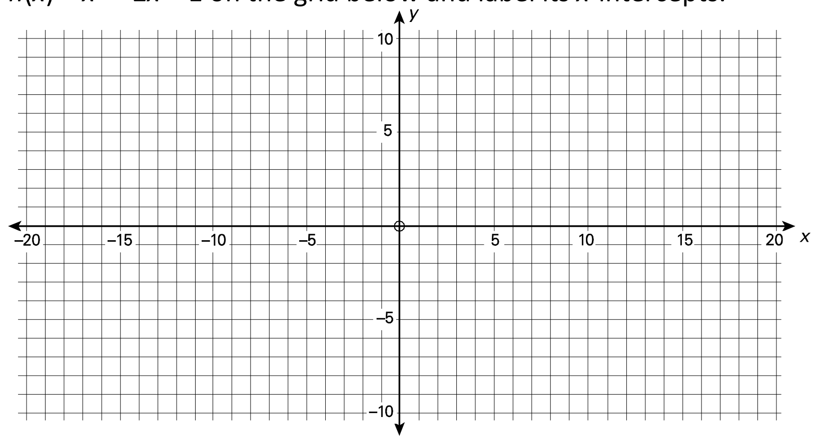
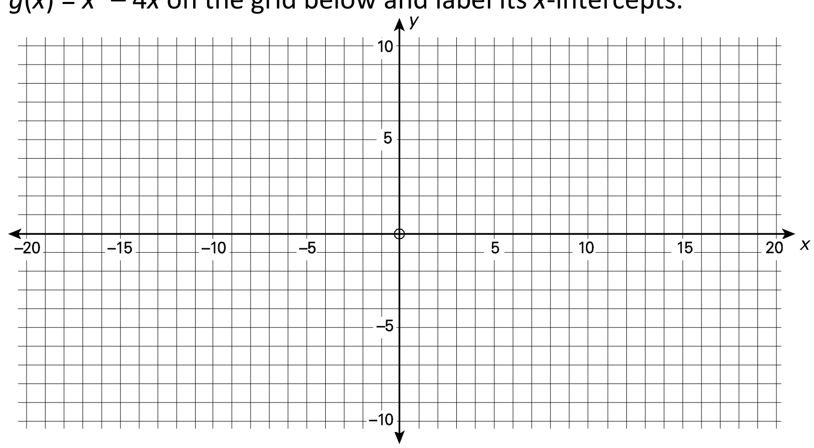
<table>
<tr><td></td><td>$h(-3)$</td><td>$h(-2)$</td><td>$h(-1)$</td><td>$h(0)$</td><td>$h(1)$</td><td>$h(2)$</td><td>$h(3)$</td><td>$h(4)$</td></tr>
<tr><td>$h(x) = x^2 - 2x + 1$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">Factored form of quadratic:</td><td colspan="7" rowspan="2">On the grid below, graph the linear equations that represent the factors of the quadratic. Use technology to sketch $h(x) = x^2 - 2x + 1$ on the grid below and label its x-intercepts.
</td></tr>
<tr><td colspan="2">State each of the linear factors of the quadratic as an equation in the form of $y = mx + b$.</td></tr>
</table>

Chart 7

<table>
<tr><td></td><td>$g(-3)$</td><td>$g(-2)$</td><td>$g(-1)$</td><td>$g(0)$</td><td>$g(1)$</td><td>$g(2)$</td><td>$g(3)$</td><td>$g(4)$</td></tr>
<tr><td>$g(x) = x^2 - 4x$</td><td></td><td></td><td></td><td></td><td></td><td></td><td></td><td></td></tr>
<tr><td colspan="2">Factored form of quadratic:</td><td colspan="7" rowspan="2">On the grid below, graph the linear equations that represent the factors of the quadratic. Use technology to sketch $g(x) = x^2 - 4x$ on the grid below and label its x-intercepts.
</td></tr>
<tr><td colspan="2">State each of the linear factors of the quadratic as an equation in the form of $y = mx + b$.</td></tr>
</table>

(b) Discuss with a partner any observations you can make based on the charts you completed in question (a). Generalize your conclusion(s).

(c) Answer the following questions:
   (i) If the factored form of $P(x)$ is $(x - 2)(x + 1)$ what are its x-intercepts?
   (ii) If $R(x)$ has x-intercepts at $x = 0$ and $x = -5$ what are its factors?
   (iii) What is the equation of a quadratic $p(x)$ if $p(-1) = 0$ and $p(-2) = 0$?
   (iv) Given that $k(a) = 0$ and $k(-a) = 0$, what do you know about the function $k(x)$?

Note: Students might conclude that when $f(-1) = 0$ they know that $f(x)$ has $(x + 1)$ as a factor and that $f(x)$ has an x-intercept at $x = -1$. They could then generalize this to when $f(-k) = 0$ we know that

$f(x)$ has $(x + k)$ as a factor and has an x-intercept at $x = -k$ OR when $f(p) = 0$ we know that $f(x)$ has $(x - p)$ as a factor and has an x-intercept at $x = p$.

- A possible task that reinforces factoring is described below.
  - For the game Three in a Row, students are given a 5 × 5 game board that consists of 25 squares. Each square will have a quadratic expression written on it. Students play in pairs. To win, a student must factor correctly three squares in a row, horizontally, vertically, or diagonally.
    - A coin is tossed to determine which student begins first.
    - The student who begins picks any square and factors the quadratic on that square. If they factor it correctly, they "win" that square.
    - The students then take turns choosing a square and factoring the quadratic on that square. Each time a quadratic is factored correctly, the student "wins" that square.
    - The first person to "win" three squares in a row wins the game.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- common factor
- difference of squares
- factor theorem
- perfect square

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 4.4, pp. 244–257

Notes

SCO RF02 Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems. [C, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF02.01 Create a table of values for $y = \left|f(x)\right|$, given a table of values for $y = f(x)$.
RF02.02 Generalize a rule for writing absolute value functions in piecewise notation.
RF02.03 Sketch the graph of $y = \left|f(x)\right|$; state the intercepts, domain, and range; and explain the strategy used.
RF02.04 Solve an absolute value equation graphically, with or without technology.
RF02.05 Solve, algebraically, an equation with a single absolute value, and verify the solution.
RF02.06 Explain why the absolute value equation $\left|f(x)\right| < 0$ has no solution.
RF02.07 Determine and correct errors in a solution to an absolute value equation.
RF02.08 Solve a problem that involves an absolute value function.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
RF01 Students will be expected to interpret and explain the relationships among data, graphs, and situations. (M10)* **RF02** Students will be expected to demonstrate an understanding of relations and functions. (M10) **RF04** Students will be expected to describe and represent linear relations, using words, ordered pairs, tables of values, graphs, and equations. (M10) **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**RF02** Students will be expected to graph and analyze absolute value functions (limited to linear and quadratic functions) to solve problems.	**RF02** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations. **RF03** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. **RF04** Students will be expected to apply translations and stretches to the graphs and equations of functions. **RF05** Students will be expected to demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the *x*-axis, *y*-axis, and line $y = x$.

* M10—Mathematics 10
** M11—Mathematics 11

Background

The concept of absolute value is new to students in this course. In Pre-calculus 11 (AN02) students have determined the absolute value of numerical expressions, and compared and ordered the absolute values of real numbers in a given set.

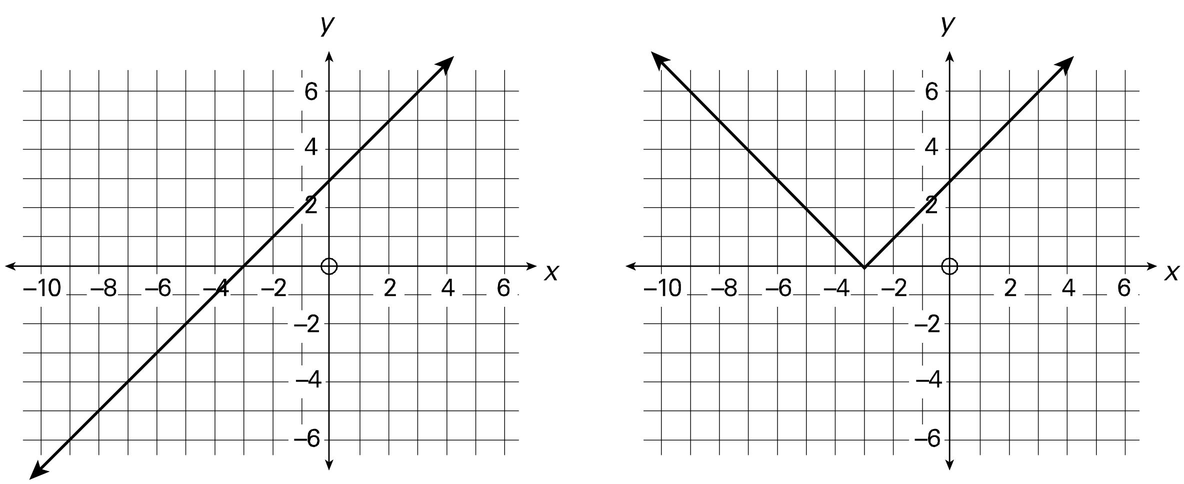
Students have been introduced to the absolute value of a number as its distance from zero on a number line. Absolute value has also been referred to as the magnitude of the number. Students have noted that the integers 5 and –5 , when plotted on a number line, are the same distance from zero and understand that absolute value only asks, How far? not, In which direction? Students have been introduced to the definition of the absolute value of any real number *a*.

$$|a| = \begin{cases} a, \text{ if } a \geq 0 \\ -a, \text{ if } a < 0 \end{cases}$$

In Mathematics 9, students solved linear equations (9PR03). In Mathematics 10, students analyzed the graphs of linear relations (10RF06). In Mathematics 11, students analyzed quadratic functions in vertex, factored, and standard form to identify the characteristics of the corresponding graph (RF02). They will now graph and analyze absolute value functions, limited to linear and quadratic functions.

Students will compare the graph of a linear function to its corresponding absolute value function, using a table of values. They will identify the similarities and differences between the table of values for the two equations $y = |ax + b|$ and $y = ax + b$ and are expected to graph $y = |ax + b|$, for example, using the table of values for $y = ax + b$.

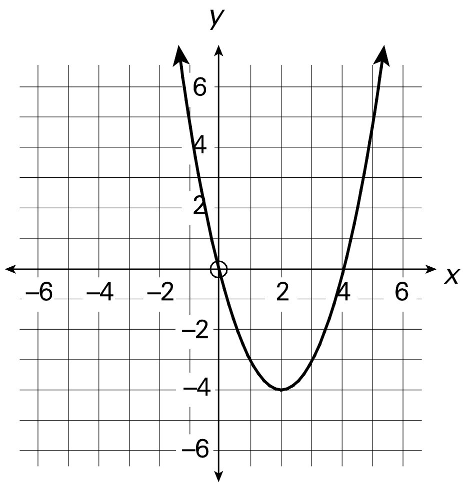
Using the graph of linear functions such as $y = x + 3$ as a visual aid to sketch $y = |x + 3|$, students should recognize and understand why the section of the graph of $y = x + 3$, which lies above the *x*-axis, remains the same, and the section that lies below the *x*-axis is reflected in the *x*-axis.

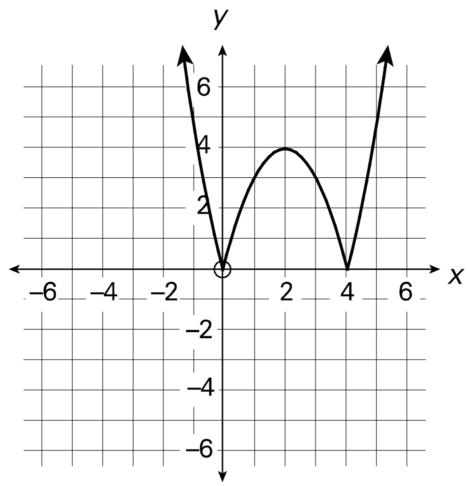


Comparing the graphs of specific linear functions such as $y = x - 1$ and $y = -2x + 6$ and the related absolute value functions $y = |x - 1|$ and $y = |-2x + 6|$ will result in the students recognizing that the absolute value function is shaped like a V and that the corner of the V occurs at the *x*-intercept of the related linear function.

Using this awareness of the shape for $y = |ax + b|$ allows students to sketch various equations of this form without the use of technology constructing a table of values.

Students will work with and analyze the absolute value of quadratic functions. They will compare the graphs of the absolute value of a quadratic function to its original graph using a table of values. Ask students to graph, for example, $y = \left|(x-2)^2 - 4\right|$ using the table of values for $y = (x-2)^2 - 4$. Students may need to be reminded that the graph of a quadratic function can be obtained using the vertex and the *x*-intercepts, as previously introduced in Mathematics 11.





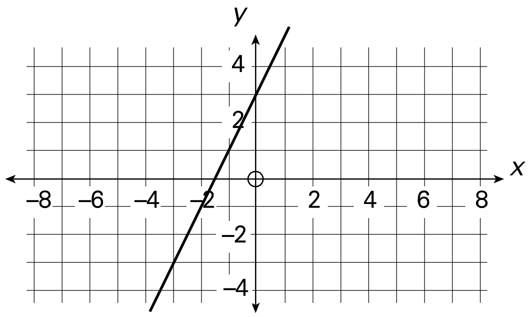
Students should notice and understand why that the graph of $y = |f(x)|$ reflects the negative part of the graph of $y = f(x)$ in the *x*-axis, while the positive part remains unchanged. Students should again note the similarities and differences between the two graphs.

Students will develop a rule for writing absolute value functions in piecewise notation using a graph.

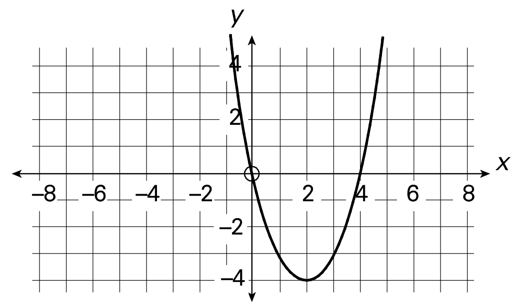
A **piecewise function** is a function composed of two or more separate functions or pieces, each with its own specific domain, that combine to define the overall function.

Students should be able to write absolute value functions such as $y = |2x + 3|$ and $y = \left|(x-2)^2 - 4\right|$ using piecewise notation. To do this they will first use the linear function $y = 2x + 3$ or the quadratic function $y = (x-2)^2 - 4$ to determine their *x*-intercepts. They will then determine the distinct intervals where the function is positive and where it is negative on a number line.

The linear function $y = 2x + 3$ is positive for values of $x > -\frac{3}{2}$, and negative for values of $x < -\frac{3}{2}$ as shown below.

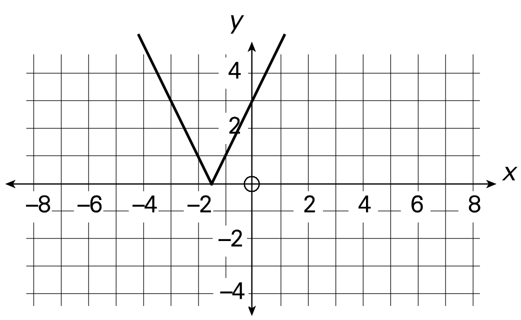


Since the quadratic $y = (x-2)^2 - 4$ is positive for values of x less than 0 and greater than 4, and $y = (x-2)^2 - 4$ is negative for values of x between 0 and 4 as shown below.



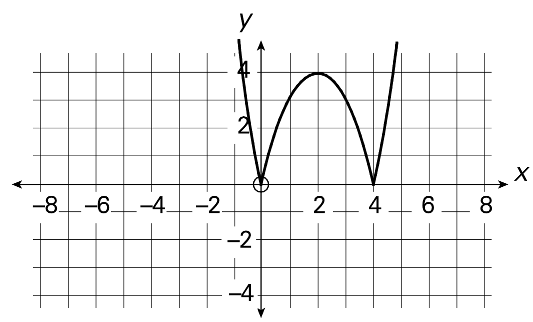
Thus, $y = \left|2x+3\right|$ could be written as

$$y = \begin{cases} 2x+3, \text{ if } x \geq -\frac{3}{2} \\ -(2x+3), \text{ if } x < -\frac{3}{2} \end{cases}.$$

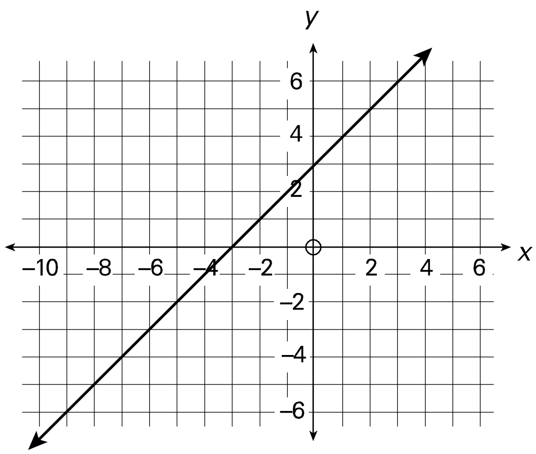


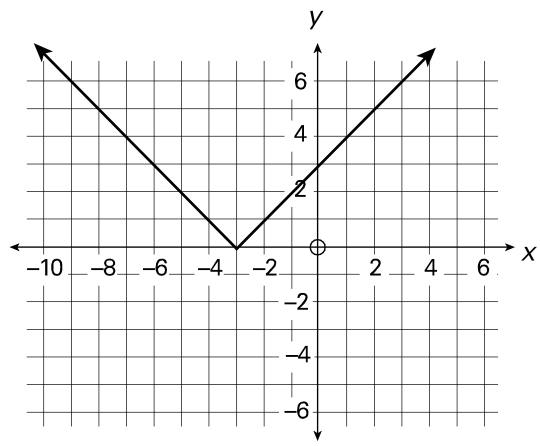
Thus, $y = \left|(x-2)^2 - 4\right|$ could be written as

$$y = \begin{cases} (x-2)^2 - 4, \text{ if } x \leq 0 \\ -((x-2)^2 - 4), \text{ if } 0 < x < 4 \\ (x-2)^2 - 4, \text{ if } x \geq 4 \end{cases}.$$



It is important to explain to students that piecewise functions are used to describe functions that contain distinct functions over different intervals. The graph of a linear absolute value function, for example, consists of two separate linear functions. The domain for each of the two intervals is dependent on the *x*-intercept(s). This can be clarified by revisiting the graphs of $y = x + 3$ and $y = |x+3|$ as shown below.





Students should notice the change in slope on the absolute value graph occurs at the *x*-intercept. For values of *x* where *y* is positive, the graphs of $y = |x+3|$ and $y = x + 3$ are the same. For values of *x* where *y* is negative, the graph of $y = |x+3|$ is a reflection in the *x*-axis of the graph of $y = x + 3$.

The absolute value function $y = |x+3|$ can be written in piecewise notation as

$$|x+3| = \begin{cases} x+3, \text{ if } x \geq -3 \\ -(x+3), \text{ if } x < -3 \end{cases}.$$

This can be generalized to all absolute value functions:

$$|f(x)| = \begin{cases} f(x), \text{ if } f(x) \geq 0 \\ -f(x), \text{ if } f(x) < 0 \end{cases}$$

Similarly, students can use the visual representation of the absolute value of a quadratic function to determine the piecewise function. This would be a good opportunity to also expose students to an algebraic method using **sign diagrams**, Figure 1, to analyze where the quadratic function, Figure 2, is positive or negative.

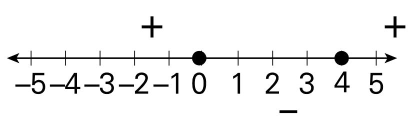


Figure 1

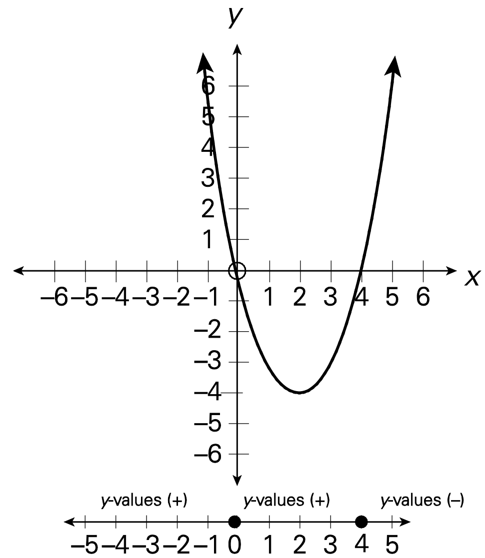


Figure 2

Using the sign diagram (Figure 1), students can write the piecewise function $y=(x-2)^2-4$ as

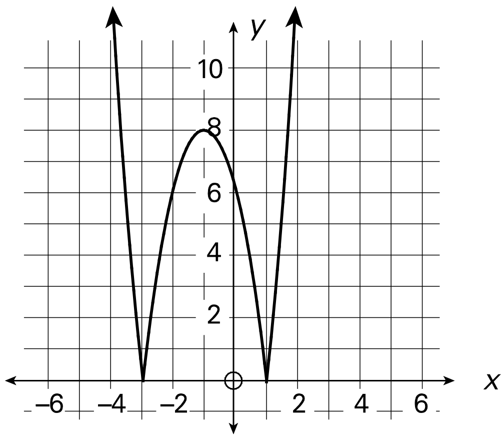
$$f(x)=\begin{cases}(x-2)^2-4, \text{ if } x\le 0, x\ge 4\\ -((x-2)^2-4), \text{ if } 0<x<4\end{cases}.$$

Alternatively, writing the absolute value function as a piecewise function can be done without the graph as a visual aid. Relating the sign diagram to the *x*-axis of the graph, students could substitute an *x*-value from each interval into the function to determine where the function is positive or negative. While this technique of testing values in each interval to obtain the sign diagram is not likely to be used by students for the absolute value of either a linear or quadratic function, it is a method that can be very useful when sketching more complicated functions whose graphs are not as likely to be easily predicted.

When graphing the absolute value of a linear or quadratic function, students should recognize the shape of the graph, the *y*-intercept, and the *x*-intercept(s). Using the information from the sign diagram, students can then draw the graph of the absolute value function.

Students will be able to work from the graph of the absolute value of either a quadratic of linear function to its equation expressed as an absolute value or as a piecewise function.

For example:



The quadratic could be written in vertex form as $y = \pm(2(x+1)^2 + 8)$ or in factored form as $y = \pm 2(x+3)(x-1)$.

The original quadratic could have been opening up **or** down since the absolute value would ensure that it was reflected in the *x*-axis so that *y*-values are all greater than or equal to zero.

Therefore the equation of the absolute value function graphed to the right could be written as $y = |2(x+3)(x-1)|$ or $y = |-2(x+3)(x-1)|$ or $y = |2(x+1)^2 + 8|$ or $y = |-(2(x+1)^2 + 8)|$

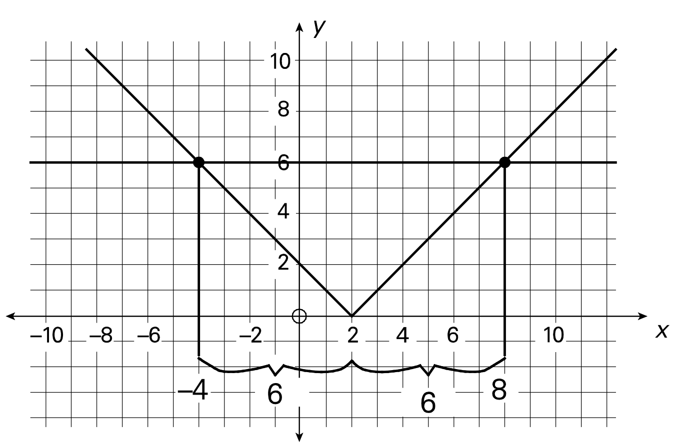
As a piecewise function it could be written as

$$y = \begin{cases} 2(x+3)(x-1), \text{ if } x \le -3, x \ge 1 \\ -2(x+3)(x-1), \text{ if } -3 < x < 1 \end{cases} \quad \text{or} \quad y = \begin{cases} 2(x+1)^2 + 8, \text{ if } x \le -3, x \ge 1 \\ -(2(x+1)^2 + 8), \text{ if } -3 < x < 1 \end{cases}.$$

Once students have an understanding of the graphs of absolute value functions, they will explore the solutions to absolute value equations, focusing first on the graphical representation and then moving to an algebraic solution. Remind them of the properties of absolute value.

- For a real number *a*, the absolute value $|a|$ is the distance from a to the origin.
- For two real numbers *a* and *b*, $|a-b|$ is the distance between *a* and *b* on the number line.

Students should first work with the absolute value of a linear equation before moving on to the absolute value of a quadratic equation. Ask students what it means to solve an equation such as $|x-2| = 6$. They should be looking for values of *x* whose distance from 2 is 6. Using a number line, students will understand that both –4 and 8 are at a distance of 6 from 2. This reasoning will allow students to better understand the solutions when using a graph.



On the graph that shows the intersection of the functions $y = |x-2|$ and *y* = 6, the intersection points, which are the values of *x* that are 6 units away from 2, are *x* = 8 and *x* = –4.

Students will move from the graphical representation of a solution to an algebraic method to solve absolute value equations. It is important to begin with simpler absolute value equations and then move on to more complex equations.

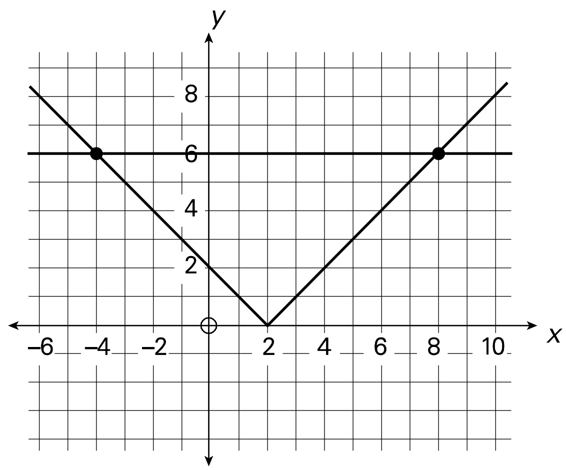
Remind students that the definition of an absolute value consists of two parts: where $f(x) \geq 0$ and $f(x) < 0$.

Consider the previous example $|x-2| = 6$. The equation is split into two possible cases: $\pm(x-2) = 6$. Students will solve the equations $(x – 2) = 6$ and $–(x – 2) = 6$ to result in $x = 8$ and $x = –4$. Each solution should be checked for extraneous roots. It is helpful to verify the solution using the intersection point(s) of the graph of the absolute value equation, $y = |x-2|$, and the graph of $y = 6$.

$$|x-2| = 6$$

$$|x-2| = 6$$

$\swarrow$	$\searrow$
$x-2 \geq 0$	$x-2 < 0$
$\downarrow$	$\downarrow$
$(x-2) = 6$	$-(x-2) = 6$
$x = 8$	$x-2 = -6$
	$x = -4$



When students solve absolute value equations involving a quadratic expression, such as $|x^2 - 4| = 3x$, they may have to solve the resulting quadratic equation using their factoring skills or the quadratic formula. The solutions obtained must be verified by substituting the x-values back into the equation and making sure the left-hand side of the equation is equal to the right-hand side of the equation.

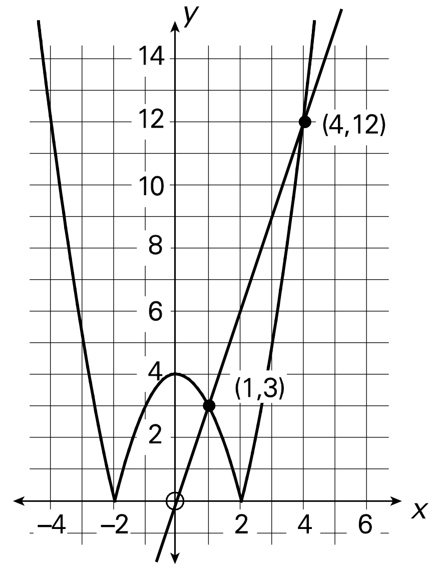
Graphically we could verify the solution(s) by considering the intersection point(s) of $y = \left|x^2 - 4\right|$ and y = 3*x*.

$$\left|x^2 - 4\right| = 3x$$

$$\left|x^2 - 4\right| = 3x$$

$$\swarrow \qquad \searrow$$

$x^2 - 4 \geq 0$	$x^2 - 4 < 0$
$\downarrow$	$\downarrow$
$x^2 - 4 = 3x$	$-(x^2 - 4) = 3x$
$x^2 - 3x - 4 = 0$	$x^2 - 4 = -3x$
$(x-4)(x+1) = 0$	$x^2 + 3x - 4 = 0$
$x - 4 = 0 \quad x + 1 = 0$	$(x+4)(x-1) = 0$
$x = 4 \qquad x = -1$	$x + 4 = 0 \quad x - 1 = 0$
	$x = -4 \qquad x = 1$



Check

$x = 4$ $\quad \left|16 - 4\right| = 12$ (*yes*)

$x = -1$ $\quad \left|1 - 4\right| = -3$ (*no*)

Check

$x = -4$ $\quad \left|16 - 4\right| = -12$ (*no*)

$x = 1$ $\quad \left|1 - 4\right| = 3$ (*yes*)

Problem solving using linear absolute value functions should be embedded throughout the unit and situated in a variety of contexts. In some problems, students will be given an absolute value function and asked to analyze it. In other problems, students will be required to create the function from the given information. Encourage students to check their answers and identify extraneous solutions.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Determine the *x*-intercept(s) and *y*-intercept of $y = (x - 2)^2 - 9$.

- Sketch $y = -2x + 3$.
- Sketch $y = -2(x - 1)(x + 3)$ labelling the x-intercept(s) and y-intercept.
- Evaluate $2\left|3(5)-20\right|-19$.

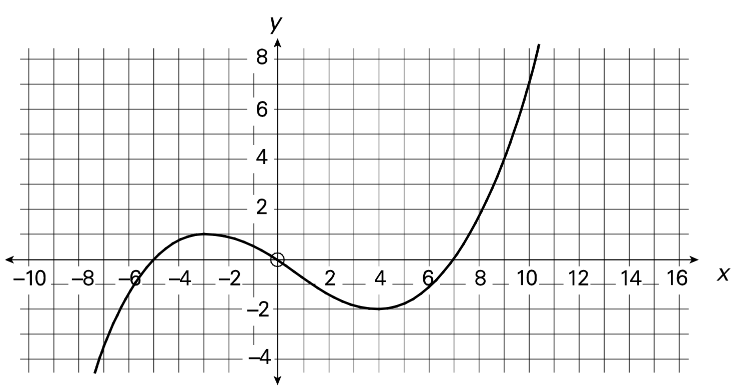
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- The following table of values is given for $y = f(x)$. Fill in the corresponding values for $y = \left|f(x)\right|$.

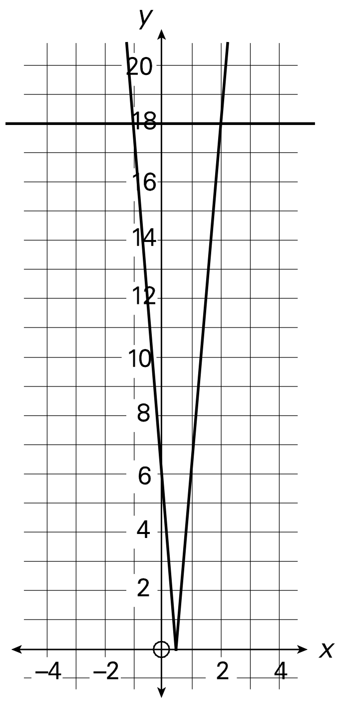
x	$f(x)$	$\left\|f(x)\right\|$
–3	32	
–2	12	
–1	–2	
0	–10	
1	–12	
2	8	
3	2	

- The graph of $g(x)$ is shown below. Sketch $\left|g(x)\right|$ on the same set of axes.



- Write $y = \left|3x + 12\right|$ as a piecewise function.
- Write $y = \left|x^2 - 3x - 10\right|$ as a piecewise function.
- Explain why the equations $y = \left|2x - 3\right|$ and $y = \left|-(2x - 3)\right|$ have the same graph.
- Write $y = \begin{cases} 2x - 6, \text{ if } x \geq 3 \\ -(2x - 6), \text{ if } x < 3 \end{cases}$ as a absolute value function.
- Explain why $\left|x + 9\right| = -2$ has no solution.

- Illustrate with a graph why $|x-2|=-6$ has no solution.
- Sketch $|h(x)|$ if you know that $h(x)$ is quadratic and that $h(x)\geq 0$ for [–1, 4].
- Maria solved $|-12x+6|=18$ algebraically and found that there were no solutions, but when she graphed the equations $y=|-2x+6|$ and y = 18, she noticed that they had two intersection points as shown below. Find and correct any errors in Maria's solution.



Solve $\lvert -12x+6\rvert=18$.	
Case 1: (when $x\geq 0$)	*Case 2:* (when x < 0)
$-12x + 6 = 18$ $-12x = 12$ $x = -1$ This is not a solution since it does not satisfy the condition that $x\geq 0$.	$-(-12x + 6) = 18$ $12x - 6 = 18$ $12x = 24$ $x = 2$ This is not a solution since it does not satisfy the condition that $x < 0$.
Therefore, there are no solutions to the equation $\lvert -12x+6\rvert=18$.	

- For what value(s) of b does $y = x + b$ not intersect the graph of $y=|x-2|$?
- For what value(s) of b does $y = x + b$ intersect the graph of $y=|x-2|$ at exactly one point?

- For what value(s) of b does the graph of $y = x + b$ and the graph of $y = \left|x^2 - 9\right|$
  (a) not intersect
  (b) intersect at exactly one point
  (c) intersect at exactly two points
  (d) intersect at exactly four points

- Determine the value of k for which the graph of $y = \left|x^2 - kx\right|$ and the graph of $y = 2x + 7$ intersect at points where $x = 7$ and $x = -1$.

- A manufacturer has a tolerance of 10 g for a bag of raisins that is supposed to weigh 500 g. Write an absolute value inequality that describes the acceptable weights for 500-g bags.

- Research a problem that incorporates the absolute value function. Be prepared to present the problem to the class, including the solution.

- What is the maximum and minimum number of possible solutions for each of the following? Explain.
  (a) $\left|ax + b\right| = 4$
  (b) $\left|ax + b\right| = -4$
  (c) $\left|ax + b\right| = 0$
  (d) $\left|ax + b\right| = 2x - 1$
  (e) $\left|ax^2 + bx + c\right| = 2$
  (f) $\left|ax^2 + bx + c\right| = 2x - 1$

- Sketch the graph of each of the following absolute value functions.
  (a) $y = \left|x + 2\right|$
  (b) $y = \left|3 - \frac{3}{2}x\right|$
  (c) $y = \left|x^2 - 4\right|$
  (d) $y = \left|3x - 3x^2\right|$
  (e) $f(x) = \left|\frac{1}{2}(x-4)(x+3)\right|$
  (f) $g(x) = \left|\frac{1}{2}(x-3)^2 - 8\right|$

- State the intercepts, domain, and range for each of the following absolute value functions.
  (a) $f(x) = \left|6 - x\right|$
  (b) $y = \left|x^2 - 4x + 3)\right|$
  (c) $y = \left|x^2 - 4\right|$
  (d) $y = \left|x^2 + 3\right|$

(e) $f(x)=\left|\frac{1}{2}(x-3)^2-8\right|$

(f) $h(x)=\left|(x-2)(x+4)\right|$

- Express each of the following absolute value functions as a piecewise function.
  (a) $h(x)=\left|x+5\right|$
  (b) $y=\left|4x-2\right|$
  (c) $y=\left|x^2-2\right|$
  (d) $k(x)=\left|(x-1)^2-3\right|$

- Algebraically solve each of the following absolute value equations.
  (a) $\left|2x-3\right|=5$
  (b) $\left|\frac{2x+1}{3}\right|=5$
  (c) $\left|x-4\right|=2x+1$
  (d) $\left|x^2+4\right|=4x$
  (e) $\left|x^2+5x-6\right|=4x$

- Algebraically determine any points of intersection of $f(x)=\left|2x\right|$ and $g(x) = 3 - x$. Use a graph to verify the solution.

- State a function that would describe each of the following graphs.

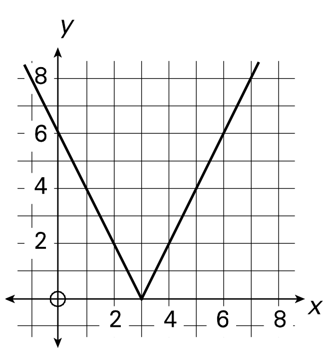


Figure 1

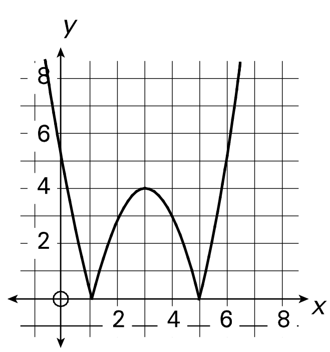


Figure 2

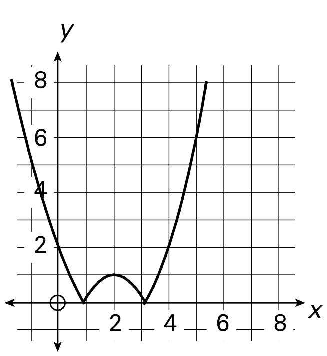


Figure 3

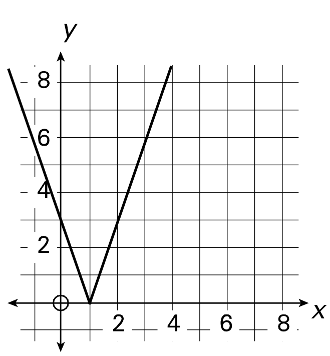


Figure 4

- The distance from Earth to the sun changes at different times of the year. The maximum and minimum distances from Earth to the sun can be represented by the equation $\left|d-149.5\right|=2.5$, where *d* is measured in millions of kilometres. Solve the equation to find the maximum and minimum distances from Earth to the sun.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Have students explore graphs of absolute functions with a graphing utility. Encourage them to predict graphs before creating the graph.

- To ensure that students understand the impact of the absolute value on a function,
  - give students a table of values for $y = f(x)$ and then have them fill in values for $\left|f(x)\right|$.
  - give students a graph for $y = f(x)$ and then have them sketch $\left|f(x)\right|$.

- Ask students to identify characteristics of the absolute value graph, such as the intercepts, domain, and range. As they look for the similarities and differences between the graphs of $f(x)$ and $\left|f(x)\right|$, they should consider the following:
  - Is the *x*-intercept(s) significant?
  - Is the domain dependent on the *x*-intercept(s)? Explain.
  - Why is the domain of the function the same as the domain of the absolute value function? Why is the range different?

- When algebraically solving an absolute value equation, students will verify their solutions graphically or by the use of substitution.
  - Students should notice the *x*-coordinates of the points of intersection are the solutions to the equation. These intersection points can sometimes be difficult to read.
  - Encourage students to verify the solution by substituting the values back into the equation.

- Provide students with worked solutions containing errors to a number of absolute value equations and asked to identify and correct the errors. Common errors include the following:
  - Treating the absolute value sign like parentheses.
  - Multiplying a constant by the expression within the absolute value sign.
  - For example, $-2|x-3| = |-2x+6|$.
  - Incorrectly placing the negative in front of the variable rather than the entire expression (e.g., When solving $|x-3|=8$, students may write $-x - 3 = 8$ instead of $-(x-3) = 8$.
  - Not identifying extraneous roots.

- It is very important that students organize their work when solving an absolute value equation. Many students will find it easiest to organize their solutions using a chart such as shown below.

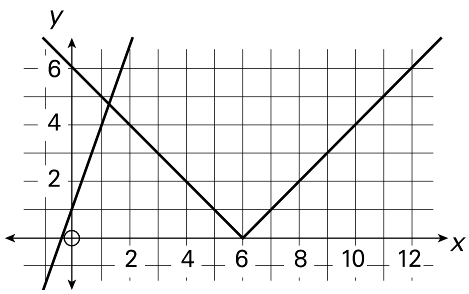
$$|x-6| = 3x+1$$

$$\swarrow \qquad \searrow$$

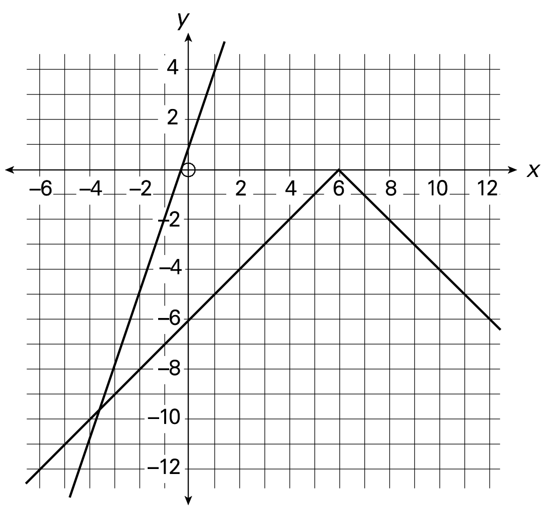
$x-6 \geq 0$	$x-6<0$
$\downarrow$	$\downarrow$
$x-6=3x+1$	$-(x-6)=3x+1$
$-7=2x$	$-x+6=3x+1$
$-\frac{7}{2}=x$	$5-4x$
	$\frac{5}{4}=x$

Check	**Check**
$\left\lvert -\frac{7}{2}-6 \right\rvert = 3\left(\frac{-7}{2}\right)+1$	$\left\lvert \frac{5}{4}-6 \right\rvert = 3\left(\frac{5}{4}\right)+1$
9.5×-95	$4.75=4.75$
(*no*)	(*yes*)
or since	or since
$-\frac{7}{2}-6 \ngeq 0$	$\left(\frac{5}{4}\right)-6<0$
(*no*)	(*yes*)

After solving an absolute value equation, ensure that students check their solutions to determine whether there are any extraneous roots. They can check their solutions by substituting into the original equation or by checking to make sure the condition stated for the specific solution has been met. They could also verify their solutions graphically. In this case there was only one solution or one intersection point. This can be verified by graphing the functions $y=|x-6|$ and $y=3x+1$. From this graph it is evident that $x=\frac{5}{4}$ is a solution while $x=-\frac{7}{2}$ is extraneous.



The extraneous solution $x=-\frac{7}{2}$ is the solution to the equation $-|x-6|=3x+1$ and can be seen graphically as the intersection point of the graphs $y=-|x-6|$ and $y=3x+1$.



- Place cards on the classroom wall consisting of linear and quadratic absolute value equations. Ask students to choose two cards, one linear and one quadratic. They should solve the equations and pass them in as exit cards. These exit cards can then be used later for students to determine and correct any errors.

- A few possible tasks that reinforce piecewise functions and absolute value functions are described below.

- In small groups, ask students to participate in a card game, Piecewise Pairs, involving graphs and piecewise functions. Each group will be given several cards. Half the cards contain a graph, while the other half will have a piecewise function. The object of the game is to be the first group to match each graph with its correct piecewise notation.

Note: Graphs should be similar enough to each other so that students cannot just match them without any thought (e.g., have several graphs with the same *x*-intercept so additional analysis is involved).

To construct this game:

- Pass one page cardstock, divided into four rows of two columns as shown to the right, to each pair of students.
- Ask student pairs to place the graphs of the absolute value of a linear equation or quadratic of piecewise function in each of the boxes on the left side of the cardstock.
- In the corresponding boxes on the right side of the cardstock, they should place the piecewise function describing the graph they have drawn.
- Photocopy these student-created cards on different colours of cardstock so that, once the cards are cut, you will have a blue deck of cards, a green deck of cards, etc.

- Working in small groups, have students play the Absolute Matching Game. Each group will be given several cards. Half of the cards will have the graph of $y = f(x)$, while the other half will have the graph of $y = |f(x)|$. The object of the game is to be the first group to pair up each graph with its absolute value graph.

  Note: For this game to be most effective, the graphs should have similar characteristics, such as the same *x*-intercepts but different slopes/vertices. To construct this game, follow the procedure outlined for the game Piecewise Pairs.

- For the task Pass the Problem, each group of students is given absolute value equations to solve algebraically. After a specific amount of time, ask students to swap their problem with another pair. If the group finished the problem, the other group will check the solution. If errors are identified, the group will correct the error and then continue to complete the problem. If the group, however, did not finish answering the problem, the other pair will check the partially completed solution and pick up from where the group left off. When they are finished, they should share the completed responses with each other, defending their reasons for any changes they made and provide feedback on each other's thinking.

SUGGESTED MODELS AND MANIPULATIVES

- cardstock
- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- absolute value
- absolute value equation
- absolute value function
- piecewise function
- sign diagrams

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 7.2 and 7.3, pp. 368–391

Notes

SCO RF03 Students will be expected to analyze quadratic functions of the form $y = a(x-p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, x-intercept and y-intercept. [CN, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF03.01 Explain why a function given in the form $y = a(x-p)^2 + q$ is a quadratic function.
RF03.02 Compare the graphs of a set of functions of the form $y = ax^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of a.
RF03.03 Compare the graphs of a set of functions of the form $y = x^2 + q$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of q.
RF03.04 Compare the graphs of a set of functions of the form $y = (x - p)^2$ to the graph of $y = x^2$, and generalize, using inductive reasoning, a rule about the effect of p.
RF03.05 Determine the coordinates of the vertex for a quadratic function of the form $y = a(x-p)^2 + q$, and verify with or without technology.
RF03.06 Generalize, using inductive reasoning, a rule for determining the coordinates of the vertex for quadratic functions of the form $y = a(x-p)^2 + q$.
RF03.07 Sketch the graph of $y = a(x-p)^2 + q$, using transformations, and identify the vertex, domain and range, direction of opening, axis of symmetry, and x- and y-intercepts.
RF03.08 Explain, using examples, how the values of a and q may be used to determine whether a quadratic function has zero, one, or two x-intercepts.
RF03.09 Write a quadratic function in the form $y = a(x-p)^2 + q$ for a given graph or a set of characteristics of a graph.

This outcome will focus on quadratic functions written in the vertex form, $f(x) = a(x-h)^2 + k$. Note that some of the performance indicators for this outcome have been greyed out. This is because they were addressed in Mathematics 11, and it is the intent of this course to extend and deepen student understanding of these performance indicators.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
RF02 Students will be expected to demonstrate an understanding of relations and functions. (M10)* **RF05** Students will be expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. (M10)	**RF03** Students will be expected to analyze quadratic functions of the form $y = a(x - p)^2 + q$ and determine the vertex, domain and range, direction of opening, axis of symmetry, x-intercept and y-intercept.	**RF02** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical translations on the graphs of functions and their related equations.

Mathematics 10 / Mathematics 11 (continued)	**Pre-calculus 11**	**Pre-calculus 12** (continued)
RF06 Students will be expected to relate linear relations expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + Bx + C = 0$) ▪ slope-point form ($y - y_1) = m(x - x_1$). (M10) **RF07** Students will be expected to determine the equation of a linear relation to solve problems, given a graph, a point and the slope, two points, and a point and the equation of a parallel or perpendicular line. (M10) **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**		**RF03** Students will be expected to demonstrate an understanding of the effects of horizontal and vertical stretches on the graphs of functions and their related equations. **RF04** Students will be expected to apply translations and stretches to the graphs and equations of functions. **RF05** Students will be expected to demonstrate an understanding of the effects of reflections on the graphs of functions and their related equations, including reflections through the x-axis, y-axis, and line $y = x$. **RF12** Students will be expected to graph and analyze polynomial functions.

* M10—Mathematics 10

** M11—Mathematics 11

Background

In Mathematics 10, students were introduced to functional notation through work with linear functions. They were introduced to the terms **relation** and **function** and determined if a relation was a function (RF02). They also determined the domain value of a linear function given a range value and vice versa (RF08). The domain and range of a graph was written using interval notation or set notation (RF01).

In Mathematics 10, students graphed linear relations by plotting the x-and y-intercepts and found the domain and range of various relations (RF08 and RF01).

In Mathematics 11 (RF02) students worked with quadratic functions in the following forms and learned how to determine vertex, intercepts, domain and range, and axis of symmetry.

Standard Form: $f(x) = ax^2 + bx + c$

Factored Form: $f(x) = a(x - r)(x - s)$

Vertex Form: $f(x) = a(x - h)^2 + k$

In Mathematics 11 (RF02), the vertex form used in the core resource was $f(x)=a(x-h)^2+k$. In this course, the vertex form used in the core resource is $f(x)=a(x-p)^2+q$. For the purposes of this background the form used will be $f(x)=a(x-p)^2+q$.

In Mathematics 11 (RF02), students worked with the vertex form of a quadratic and were able to

- sketch the quadratic, labelling the vertex and the *y*-intercept

- identify the number of *x*-intercepts for the quadratic function, $f(x)=a(x-p)^2+q$, from the values of *a* and *p*
  - If *a* is positive, the parabola opens upward; if negative, it opens downward. If *a* and *q* are both positive, or if they are both negative, the parabola will either open upward with a vertex above the *x*-axis or will open downward with a vertex below the *x*-axis, and there will be zero *x*-intercepts.
  - If *a* and *q* have opposite signs, the parabola will open upwards with a vertex below the *x*-axis, or will open downward with the vertex above the *x*-axis, and there will be two *x*-intercepts.
  - If *a* is either positive or negative but *q* = 0, the vertex will sit right on the *x*-axis, and there will be only one value for the *x*-intercept. Students will be aware that the value of *a* indicates the opening direction of the parabola.

- Given a graph of a quadratic function, students will determine the quadratic function in the form $f(x)=a(x-p)^2+q$, determine the vertex directly from the graph, and, hence, the values of *p* and *q*. When determining the value of *a*, students can substitute a point (*x*, *y*) that is on the parabola into the quadratic function and algebraically solve for *a*. Students have *not* used the language stretch or translation.

- Given the vertex, as a maximum or minimum value, and at least one other point, students used substitution to determine the quadratic function in vertex form.

- Use the vertex form of a quadratic function to solve for the independent variable given a value for the dependent variable and vice versa.

- Students should recognize that all non-contextual quadratic functions have a domain of $\{x \mid x \in R\}$ whereas the range depends on the vertex and the direction of opening. Students will understand that a negative value of *a* will indicate that the range is less than or equal to *q* while a positive value of *a* will indicate that the range is greater than or equal to *q*.

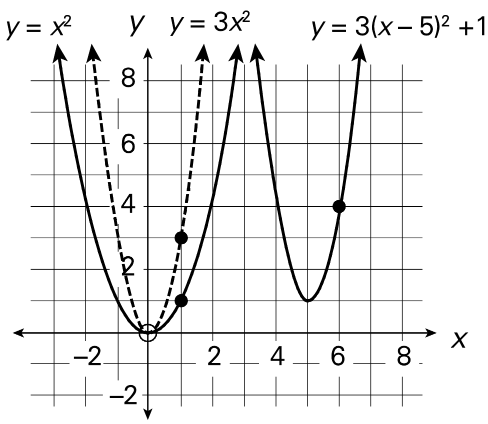
In Pre-calculus 11, students will investigate the equation $f(x)=a(x-p)^2+q$ and the effect of changing its parameters *a*, *p*, and *q*, in terms of vertical and horizontal transformations.

Students began by investigating changes in the value of *a* by comparing quadratic functions $y = x^2$ and $y = ax^2$. Students will know what happens to the direction of opening of the quadratic if $a < 0$ and $a > 0$ and that the shape of the parabola is affected by the parameter *a*. (Mathematics 11 RF02)

The focus of this course is to develop the specific impact of the value of *a* as a vertical stretch factor. As they compare the graphs of $y = 2x^2$ and $y = -2x^2$ to the graph of $y = x^2$, encourage students to pay particular attention to the points and their corresponding points and how they change. For example, the point (2, 4) on the graph of $y = x^2$ corresponds to the point (2, 8) on the graph of $y = 2x^2$.

Similarly, students can compare the graphs of $y = \frac{1}{2}x^2$ and $y = -\frac{1}{2}x^2$ to the graph of $y = x^2$.

In this course, students can consolidate their learning by sketching the graph of $f(x) = a(x-p)^2 + q$ using transformations for various values of a, p, and q. They can apply the change in "width" using the value of a by selecting the vertex and two other points on the graph of $y = x^2$. They can then use the values of p and q to translate the graph. Consider the function $y = 3(x - 5)^2 + 1$. The a value, 3, results in a "narrower" graph. The points (0, 0), (–1, 1) and (1, 1) on the graph $y = x^2$ will correspond to (0, 0), (–1, 3), (1, 3) on the graph of $y = 3x^2$. Students will then translate the graph using the horizontal translation (+5 units or right 5 units) and the vertical translation (+1 unit or up 1 unit). The points transform to (5, 1), (4, 4), and (6, 4). Students should note the vertex and symmetry of the graph and other features such as the domain, range, and intercepts. It is important for students to understand that the order of transformations is R(reflections) S(stretches) T(translations).



In this course, in addition to discussion of wideness and narrowness of a quadratic based on the value of a, students will explore how the value of a also affects the **vertical stretch** or how the y-values increase in relation to the x-values. When $|a| < 1$, the y-values increase or decrease more slowly than they would for $y = x^2$ and so the graph appears "wider". When $|a| > 1$, the y-values increase or decrease more quickly and so the parabola appears "narrower" than the graph of $y = x^2$.

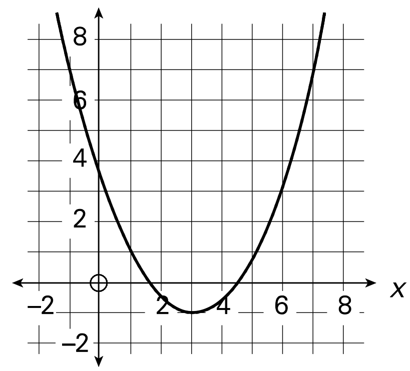
For the equation $f(x) = a(x-p)^2 + q$, the value of p, indicates the magnitude of the **horizontal translation** of $y = x^2$. The sign before p in the equation indicates the direction of the translation; if the sign is positive, the parabola is shifted p units to the left; if the sign is negative, the parabola is shifted p units to the right. For example, for $y = 3(x + 2)^2 + 6$, the parabola is shifted two units to the left.

The value of q indicates the **vertical translation** of $y = x^2$. When $q < 0$ the parabola is shifted down q units. When $q > 0$ the parabola is shifted up q units.

Because the vertex for $y = x^2$ is at (0, 0), and every point moves right or left p units and up or down q units, the vertex of the parabola will be at (p, q).

Give students sufficient time to compare equations to the corresponding graphs so they will be able to draw these conclusions.

The mapping rule is not an outcome of this course and will be addressed in Pre-calculus 12. In this course students will be expected to describe the graph of a quadratic function, as well as the equation of a quadratic that is written in vertex form, in terms of its vertical and horizontal transformations.

<table>
<tr>
<td>Example A
Given the function $y = -2(x-3)^2 + 5$</td>
<td>Example B
Given the graph, determine its equation.
</td>
</tr>
<tr>
<td>Describe in terms of transformation of the graph $y = x^2$.

Reflection in the x-axis
Vertical stretch of 2
Vertical translation of +5 (or up 5)
Horizontal translation of +3 (or right 3)

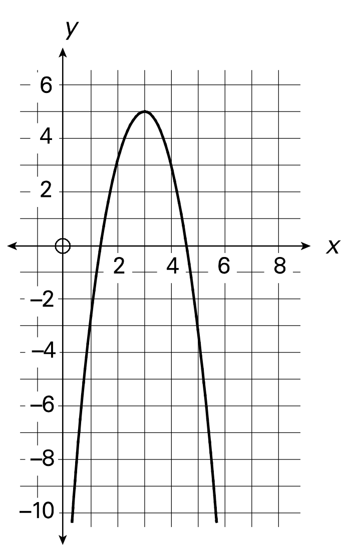
Vertex: (3, 5)

Right/left 1 down $2(1)^2 = 2$; Points (4, 3) and (2, 3)
Right/left 2 down $2(2)^2 = 8$; Points (5, –3) and (1, –3)
Right/left 3 down $2(3)^2 = 18$; Points (6, –13) and
(0, –13)
</td>
<td>Describe in terms of transformation of the graph $y = x^2$.

Horizontal translation of +3
Vertical translation of –1

From the vertex:

Right/left 1 up 0.5 (instead of $(1)^2$ or 1)
Right/left 2 up 2 (instead of $(2)^2$ or 4)
Right/left 3 up 4.5 (instead of $(3)^2$ or 9)

Therefore vertical stretch of $\frac{1}{2}$.</td>
</tr>
<tr>
<td></td>
<td>Equation: $y = \frac{1}{2}(x-3)^2 - 1$</td>
</tr>
</table>

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

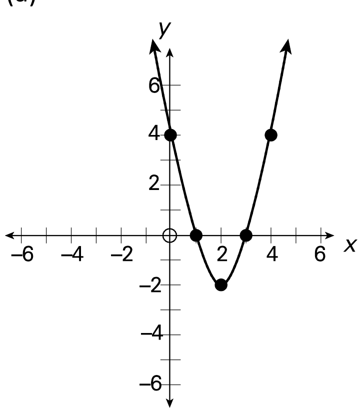
Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

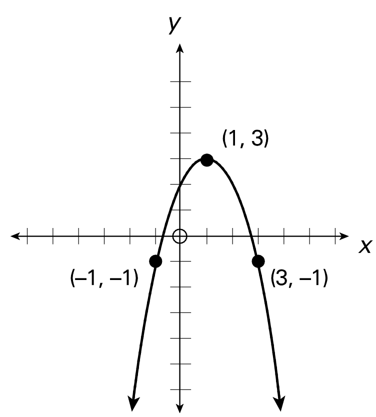
- For each of the quadratic functions below, sketch the graph and determine: vertex, domain and range, direction of opening, axis of symmetry, x-intercept and y-intercept.
  (a) $y = 3(x - 5)^2 + 4$
  (b) $y = -x^2 - 1$
  (c) $y = 12(x + 2)^2$
  (d) $y = -2(x - 4)^2 - 3$

- From the graphs of the quadratic functions below, write the equation in the form of $y = a(x - p)^2 + q$.

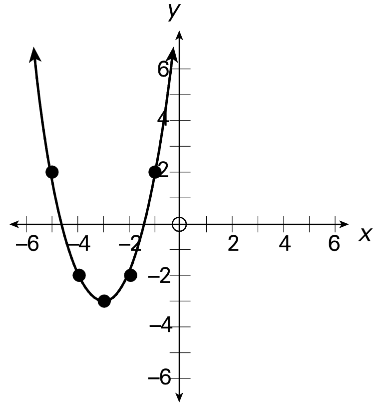
(a)



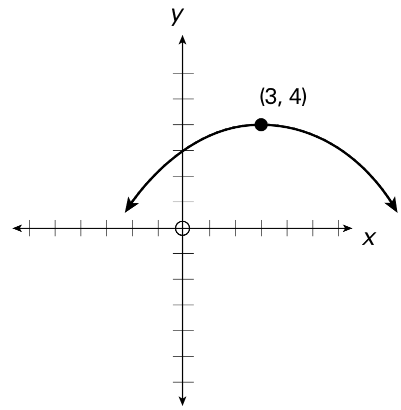
(b)



(c)



(d)



- Write the equation of the quadratic function in the form of $y = a(x - p)^2 + q$ given:
  (a) vertex (5, −7); $a = 2$
  (b) vertex (0, 3); $a = -14$
  (c) parabola opens downward but keeps its original shape (as for $y = x^2$); vertex (−2, 3)

- Determine the vertex form of the quadratic function from the given table of values.

x	–6	–5	–4	–3	–2	–1	0
y	1	3	1	–5	–15	–29	–47

- Determine the vertex form of the quadratic function given the following information.
  – Range is $\{y \mid y \le 3, y \in R\}$ and the x-intercepts are –2 and 4.
  – Equation of the axis of symmetry is $x = 2$, the minimum value of y is –5, and the y-intercept is 3.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe the reasoning used to decide whether each statement is true or false.

Polynomial Function	Classification	True or False	Explain/Justify
$y = 5(x + 3)$	linear		
$y = 5(x^2 + 3)$	quadratic		
$y = 5^2(x + 3)$	quadratic		
$y = 5x(x + 3)$	linear		
$y = (5x + 1)(x + 3)$	quadratic		
$y = 5(x + 3)^2 + 2$	quadratic		

- A parabola has vertex (4, 3) and one x-intercept at (1, 0). Find the other x-intercept.

- To solve $5x^2 + 30x - 10 = 0$ by completing the square, the first step is to divide both sides by 5. Explain why this must be done.

- Solve $x - \frac{1}{x} = 1, x \neq 0$, by completing the square.

- Let *A* and *B* be the points of intersection of $f(x) = x^2 - 8x + 7$ and the *x*-axis, and let *C* be the vertex of $f(x)$. Determine the area of ΔABC.

- Determine the value(s) of *m* for which the graph of the parabola $y = x^2 + mx + 16$ is always above the *x*-axis.

- Determine the value of *k* so that the parabola whose equation is $y = x^2 + 2x + 2k - 3$ will have its vertex on the *x*-axis.

- Describe each of the following graphs in terms of transformations of the graph of $y = x^2$.

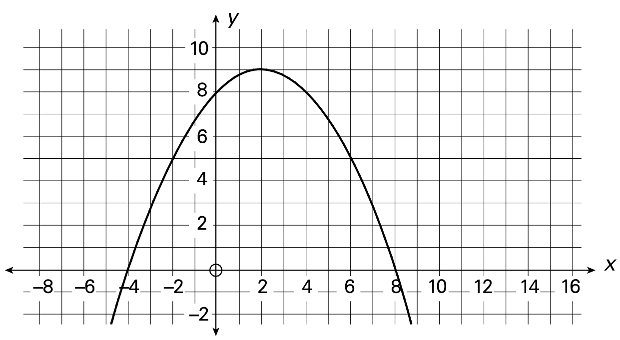


Figure 5

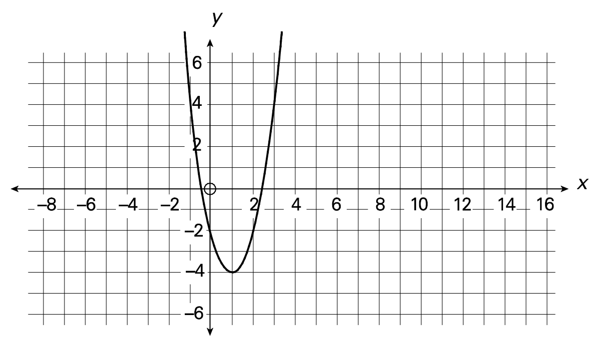


Figure 6

- Describe each of the following equations in terms of transformations of the graph of $y = x^2$.

  (a) $y = \frac{1}{4}(x-2)^2 + 5$

  (b) $y = 2(x - 3)^2$

  (c) $y = \frac{1}{2}(x+3)^2 - 4$

  (d) $y = 3x^2 + 1$

- Use transformations of $y = x^2$ to sketch each of the following graphs labelling the vertex and a minimum of four other points.

  (a) $y = -3(x - 2)^2 + 1$

  (b) $y = 3(x - 3)^2 + 1$

  (c) $y = -\frac{1}{2}x^2$

  (d) $y = \frac{1}{3}(x-2)^2$

  (e) $y = 3x^2 + 1$

- Your friend has missed the class on determining the vertex for any quadratic function when it is written in vertex form and asks you to explain how this is done. Ask students to write a paragraph, with examples, explaining how this is done.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Use technology (graphing software) to show the effects of the stretch factor a on the parabola $y = x^2$ for many positive and negative values of a. Repeat the same process separately for the horizontal shift p and the vertical shift q.

  When given the equations $y = x^2$ and $y = x^2 + 4$, for example, students make the connection that the original y-values are increasing by 4 but the x-values remain unchanged. Adding 4 to the function, therefore, means a vertical translation for every point on the graph of positive 4 or up 4.

  Students sometimes have more conceptual problems with horizontal translations. Consider the equations $y = x^2$ and $y = (x + 4)^2$.

  In Mathematics 10, students represented functions using function notation. The concept was introduced as a rule that takes an input value and gives a unique corresponding output value. You can use the idea of functional notation to explain to students that the horizontal translation is the opposite operation inside the parenthesis.

  Given: $f(x) = x^2$, we know that $f(3) = 9$ and $f(-3) = 9$.

Suppose you had $g(x) = (x + 2)^2$ for what values of x would $g(x) = 9$?

$(x+2)^2 = 9$

$(x+2) = \pm 3$

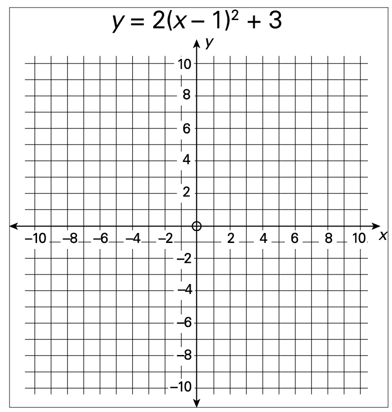
$x = -2 \pm 3$

For the function $g(x) = (x + 2)^2$ the values of x that have this same value of y are 2 units to the left of the original value of x.

- Have students discuss how the shape, position, and direction of the parabola are altered when the parameters in $y = a(x-p)^2 + q$ are changed.

- Using technology, explore the number of x-intercepts the function $y = a(x-p)^2 + q$ has in reference to the values of a and p.

- Students can work in pairs to construct the graphs of a number of quadratic functions of the form $y = a(x-p)^2 + q$, using transformations for various values of a, p, and q. Each group is provided with a handout containing a grid and the function. Place the handout inside a sheet protector so each group can graph a different function using an erasable marker. Ask questions to the group related to the characteristics of the quadratic function (vertex, direction of opening, x-intercept and y-intercept, domain and range). When completed, ask students to erase the function and pass it along to another group. Repeat the task, providing students the opportunity to construct a number of graphs.



- Give each student a card with a different equation in the form of $y = (x-p)^2$ and have students graph the function on a small grid and state the transformations of each graph. Place students in groups of six to eight to review their graphs and transformations and to correct any errors. Have students shuffle the cards of graphs and equations and exchange cards with another group who will, in turn, match equations and graphs.

  Variation on the above: Have students give the pile of graphs to the other group and have them write the equation and transformations. They would then be given the pile of original equations or transformations for the graphs they were given in order to correct their work.

  Variation on the above: Have students give the pile of transformations to the other group and have them write the equation and draw the graphs. They would then be given the pile of equations or graphs for the graphs they were given in order to correct their work.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- axis of symmetry
- domain
- horizontal translation
- maximum Value
- minimum Value
- parabola
- quadratic function
- range
- reflection in the *x*-axis
- transformation
- vertex
- vertex form
- vertical stretch
- vertical translation
- *x*-intercept
- *y*-intercept

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 3.1, pp. 142–162

  (**Note:** Many of the questions in this section will be about materials taught as part of Mathematics 11.)

Notes

SCO RF04 Students will be expected to analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including vertex, domain and range, direction of opening, axis of symmetry, *x*-intercept and *y*-intercept, and to solve problems. [CN, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF04.01 Explain the reasoning for the process of completing the square as shown in a given example.
RF04.02 Write a quadratic function given in the form $y = ax^2 + bx + c$ as a quadratic function in the form $y = a(x - p)^2 + q$ by completing the square.
RF04.03 Identify, explain, and correct errors in an example of completing the square.
RF04.04 Determine the characteristics of a quadratic function given in the form $y = ax^2 + bx + c$, and explain the strategy used.
RF04.05 Sketch the graph of a quadratic function given in the form $y = ax^2 + bx + c$.
RF04.06 Verify, with or without technology, that a quadratic function in the form $y = ax^2 + bx + c$ represents the same function as a given quadratic function in the form $y = a(x - p)^2 + q$.
RF04.07 Write a quadratic function that models a given situation, and explain any assumptions made.
RF04.08 Solve a problem, with or without technology, by analyzing a quadratic function.

Scope and Sequence

Mathematics 10 / Mathematics 11	**Pre-calculus 11**	**Pre-calculus 12**
RF01 Students will be expected to interpret and explain the relationships among data, graphs, and situations. (M10)* **RF02** Students will be expected to demonstrate an understanding of relations and functions. (M10) **RF05** Students will be expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. (M10) **RF06** Students will be expected to **r**elate linear relations to their graphs expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + By + C = 0$) ▪ slope-point form ($y - y_1) = m(x - x_1$) (M10)	**RF04** Students will be expected to analyze quadratic functions of the form $y = ax^2 + bx + c$ to identify characteristics of the corresponding graph, including vertex, domain and range, direction of opening, axis of symmetry, *x*-intercept and *y*-intercept, and to solve problems.	**RF12** Students will be expected to graph and analyze polynomial functions.

Mathematics 10 / Mathematics 11 (continued)	Pre-calculus 11	Pre-calculus 12
RF07 Students will be expected to determine the equation of a linear relation, given a graph, and a point and the slope, two points, and a point and the equation of a parallel or perpendicular line. (M10) **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**		

* M10—Mathematics 10

** M11—Mathematics 11

Background

In Mathematics 10, students factored differences of squares, perfect square trinomials, and polynomials of the form x^2+bx+c and ax^2+bx+c (10AN05).

In Mathematics 11, students examined quadratic functions expressed in standard form $y=ax^2+bx+c$ and determined the x-intercept and y-intercept, axis of symmetry, direction of opening, vertex (as a maximum or minimum point) using the symmetry of the quadratic and the formula $x=-\frac{b}{2a}$ to determine the x-coordinate of the vertex, and the domain and range. They also sketched the graph of a quadratic function using the characteristics above. Students studied the effect of changing the parameters a, b, and c of the equation on the shape of the graph and solved problems involving quadratic functions. (RF02)

For this outcome, students will be required to express quadratic functions in the form $y=a(x-h)^2+k$ or $y=a(x-p)^2+q$. This will require learning to complete the square, which they have not seen before.

In Mathematics 10, students modelled a quadratic expression using algebra tiles. They also demonstrated, through the use of algebra tiles, an understanding of the multiplication of polynomial expressions (limited to monomials, binomials, and trinomials), common factoring, and trinomial factoring (10AN04, 10AN05). Note that students have factored using positive alge-tiles only. Those patterns developed using alge-tiles were then extended to other factorable quadratics.

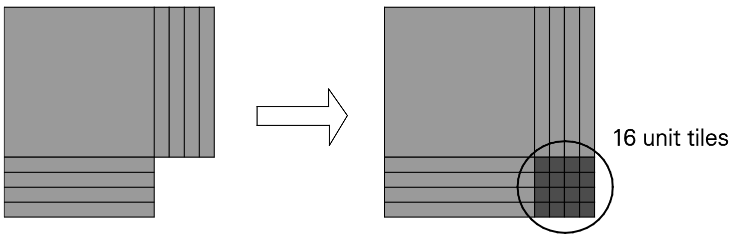
In this course, students use algebra tiles to understand the process of creating perfect square trinomials and the patterns formed. The method of completing the square is one method students will use when they algebraically rewrite a quadratic equation from standard form to vertex form. The perfect square binomial $(x-h)^2$ or $(x-p)^2$ is part of the quadratic function in vertex form.

Students should first be exposed to quadratic functions where $a = 1$ and then progress to examples where $a \neq 1$.

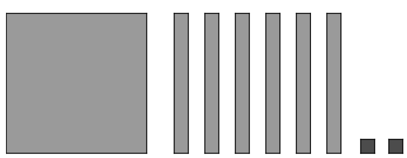
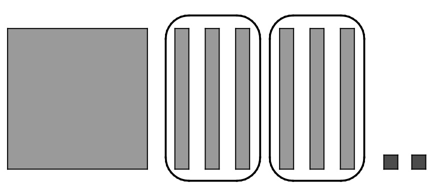
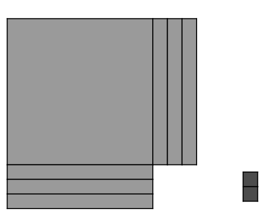
Students should be familiar with squaring binomials, and can be encouraged to identify patterns between the binomial and the resulting trinomial. For example, for $(x + 5)^2 = x^2 + 10x + 25$, the middle and the last term of the trinomial are related to the second term in the binomial $(2)(5) = 10$, $5^2 = 25$.

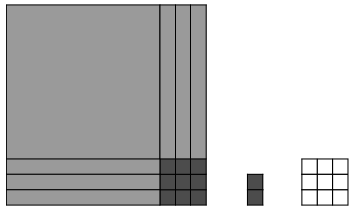
Give students the opportunity to complete the square using algebra tiles and algebraically.

Using algebra tiles
For the expression $x^2 + 8x$, the terms are arranged to form the beginning of a square that can then be completed by filling in with unit tiles in the bottom right corner.



Students can also apply this knowledge to express a quadratic equation in vertex form. For example, for $y = x^2 + 6x + 2$, one x^2 algebra tile, and six x tiles and two unit tiles are arranged as shown below. The square is completed by adding the missing +9 unit tiles in the bottom right corner. To maintain equivalency, nine negative unit tiles must also be added.

	$y = x^2 + 6x + 2$
Splitting the 6x into two equal groups of 3x each.	
	$y = x^2 + 3x + 3x + 2$
Rearranging to begin the formation of a square.	
	

Now to determine how many 1 tiles are needed to complete the square.	
	$y = (x^2 + 6x + 9) + 2 - 9$ $y = (x + 3)^2 - 7$
The dimensions of the square gives the factor $(x + 3)$ and the remaining tiles −7 represent the vertical translation. Therefore: $x^2 + 6x + 2 = (x + 3)^2 - 7$.	

Algebraically

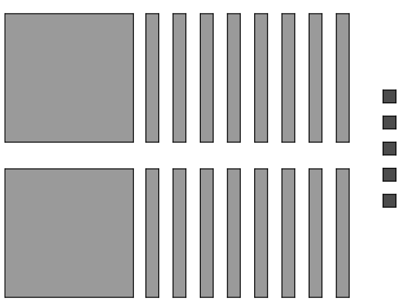
The numerical coefficient of the second term of the trinomial is halved and then that value is squared to determine the third term needed to make a perfect square. This value is added and then subtracted to maintain equivalency. The perfect square is factored and the remaining terms are combined.

For example, for $y = x^2 + 6x + 2$, half of 6 is 3, 3 squared is 9, so 9 is added and also subtracted, the perfect square is factored to $(x + 3)^2$, and the remaining terms are combined:

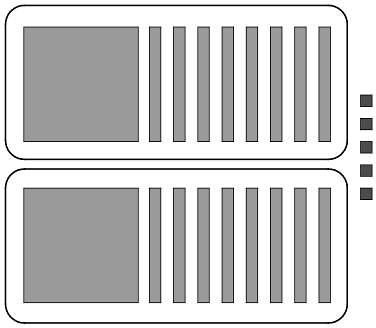
$y = x^2 + 6x + 2$
$y = x^2 + 6x + 9 - 9 + 2$
$y = (x + 3)^2 - 7$

Once the equation is in this form, the vertex and the axis of symmetry can be identified. For the example above, the vertex is (–3, –7) , and the axis of symmetry is $x = -3$.

Similarly, you can use algebra tiles to illustrate completing the square for leading coefficients other than 1.

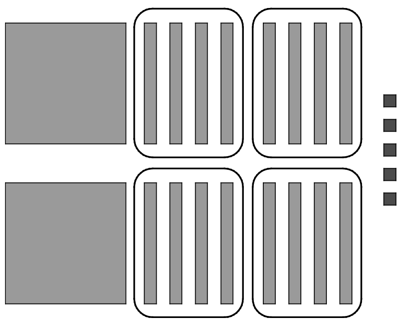
Use alge-tiles to represent the trinomial.	
	$y = 2x^2 + 16x + 5$

Split the x^2 and x tiles into two equal groups.



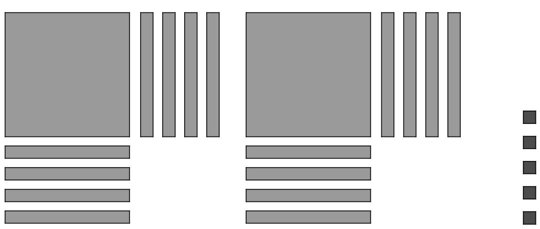
$$y = 2(x^2 + 8x) + 5$$

For each group, split the x tiles into two equal groups.

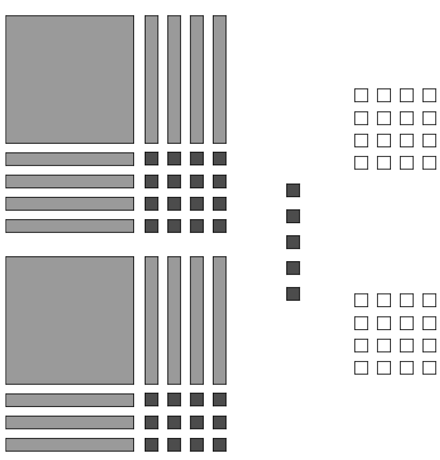


$$y = 2(x^2 + 4x + 4x) + 5$$

For each group, rearrange to begin forming squares.



Adding 16 unit tiles to each square completes the squares. To balance, add (2)(16) or 32 negative unit tiles.



$$y = 2(x^2 + 4x + 4x + 16) + 5 - 32$$
$$y = 2(x + 4)^2 - 27$$

Algebraically, if the coefficient of x^2 is not 1, the leading coefficient is factored from the first two terms before completing the square. For example, for $y = 2x^2 - 16x + 25$, the first step would be to factor 2 from $2x^2 - 16x$:

$y = 2x^2 - 16x + 25$
$y = 2(x^2 - 8x) + 25$
$y = 2(x^2 - 8x + 16 - 16) + 25$
$y = 2(x^2 - 8x + 16) - 32 + 25$ (**Note:** -16 must be multiplied by 2 when bringing it out of the parentheses.)
$y = 2(x - 4)^2 - 7$

When converting a quadratic function from vertex form to standard form, students will expand the perfect square trinomial and then combine like terms. Students should also verify that a quadratic function in standard form represents the same function in vertex form. They can use the method of completing the square to compare the functions or compare the graphs of both functions. Ask students what features of the graph must be the same. They must ensure that both functions have the same vertex and one other point (e.g., the *y*-intercept) on both graphs.

As an alternative, an algebraic approach combined with a graphical approach could be used. Ask students to graph one of the functions. They should choose three points from the graph and substitute them into the other form of the function to verify the points satisfy the function. Promote student discussion as to why three points must be used and not two points.

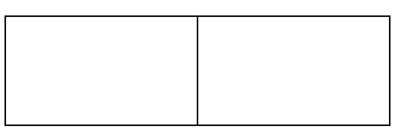
In Mathematics 11 (RF02), students used the form of the quadratic function that would most appropriately apply to the situation and solved contextual problems where

- the equation is given
- the vertex was evident and one other point was available
- the *x*-intercepts and one other point were given

In this course, students will explore problems where they will determine the maximum or minimum value. Contextual problems could involve maximum revenue, finding the maximum possible area, the maximum height, or a minimum product.

These will include contextual problems that were not covered in Mathematics 11 such as,

- You have 600 metres of fencing and a large field. You want to make a rectangular enclosure split into two equal lots. What dimensions would yield an enclosure with the largest area?



- Harmony sells bracelets for \$26. At this price she can sell 200 bracelets per month. She wants to increase her revenue, but she knows that she will sell 5 fewer bracelets per month for each increase of \$1. What should Harmony charge for her bracelets if she wants the maximum revenue?

- Two consecutive integers are squared. The sum of those squares is 365. What are the integers?

It is important for students to understand the terminology that is being used in the problems. They should make the connection between the maximum or minimum value, and the *y*-value of the vertex.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

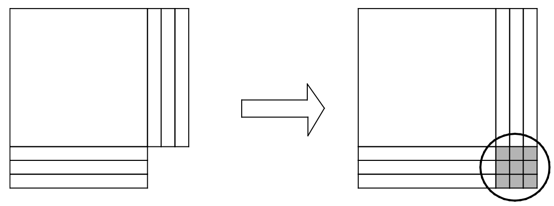
- What do you know about the graph of the function $y = -2x^2 - 6x + 1$?
- What do you know about the graph of the function $y = 3x^2 + 12x + 120$?
- Why is it to your advantage to write an equation in vertex form?
- Describe the effects of the parameters *a*, *b*, and *c* on the quadratic function $y = ax^2 + bx + c$.

Whole-Class/Group/Individual Assessment Tasks

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Use algebra tiles to demonstrate how you could complete the square given each of the following:
  (a) $x^2 + 6x$
  (b) $x^2 + 6x + 1$
  (c) $4x^2 + 8x$
  (d) $4x^2 + 8x + 12$
- Complete the square for $y = ax^2 + bx + c$ to show that the coordinates of the vertex are
  $\left(-\frac{b}{2a}, \frac{4ac-b2}{4a}\right)$

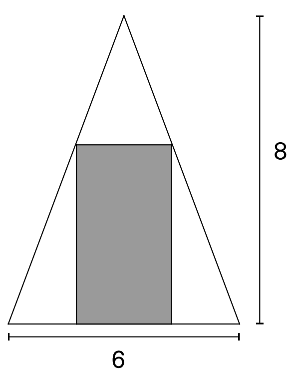
- Complete a table investigating the relationship between the value of *b* and *c* when completing the square. A sample is shown below.

Binomial of the form $x^2 + bx$	Modelled with algebra tiles	Perfect square
$x^2 + 6x$		$x^2 + 6x + 9$

What was added to complete the square?	Factored and expanded form	Value of constant	How are *b* and *c* related?
9	$(x + 3)(x + 3) =$ $x^2 + 6x + 9$	9	$c = (\frac{1}{2}b)^2$

- A stream of water from a fountain forms a parabolic shape. Given the spout on the fountain is 5 cm high and the maximum height reached by the water is 14 cm at a distance of 6 cm from the spout, what is the height of the water when it is 8 cm from the spout?
- A student makes and sells necklaces at the beach during the summer months. The material for each necklace costs her \$6 and she has been selling about 20 per day at \$10 each. She has been wondering whether or not to raise the price, so she takes a survey and finds that for every dollar increase she would lose two sales a day. What price should she set for the necklaces to maximize profit?

- Find the dimensions of the rectangle of maximum area that can be inscribed in an isosceles triangle of altitude 8 and base 6. (Hint: Use similar triangles to express the height of the rectangle in terms of its base).



- Write each of the following quadratic functions in the form $y = a(x - p)^2 + q$.
  (a) $y = x^2 - 10x + 19$
  (b) $y = 3x^2 + 18x + 22$
  (c) $y = -2x^2 - 8x - 23$
  (d) $y = -23x^2 + 12x - 31$

- Using the answers for (a) and (b) in the previous question, determine the characteristics of the quadratic functions and sketch the graphs.

- A football is kicked into the air and follows the path $h = -5x^2 + 20x$, where x is time in seconds and h is the height of the ball above the ground in metres.
  (a) Determine the characteristics of the function and sketch the graph.
  (b) What is the maximum height attained by the football?
  (c) How long does the ball stay in the air?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

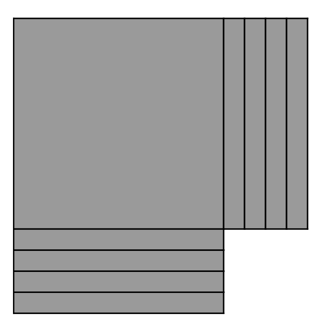
Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

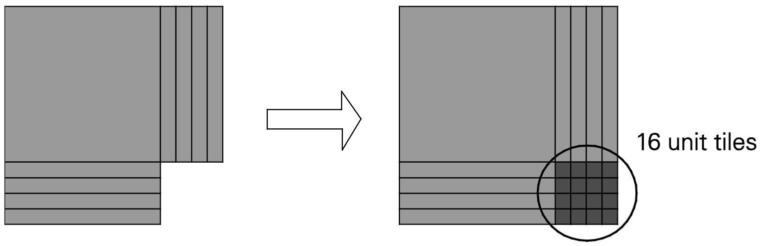
- Algebra tiles can be used to visualize how to create a perfect square trinomial can be formed. Students should first be exposed to examples where $a = 1$ and $c = 0$. Consider the following example:

  – Ask students to model $x^2 + 8x$. The goal is to find a number c to create a perfect square trinomial $x^2 + 8x + c$. Students will use the algebra tiles to create a square. Ask students why the number of x-tiles must be split evenly.



Continue to use leading questions, such as the following, to promote discussion.

- What tiles must be added to complete the square?



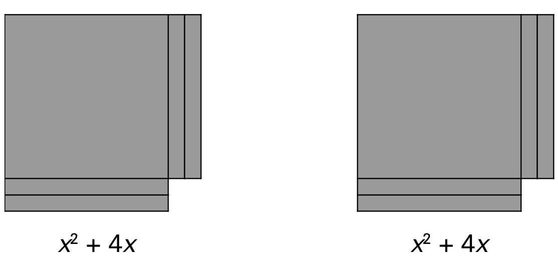
- What is the expression that represents the new completed square?
- What is the relationship between the coefficient of the linear term and the constant term?
- What is the trinomial written as the square of a binomial?

Students should understand that 16 unit tiles have been added. Since the side length of the square is represented by $(x + 4)$, the area is $(x + 4)(x + 4)$. This perfect square trinomial $x^2 + 8x + 16$, can be rewritten as $(x + 4)^2$.

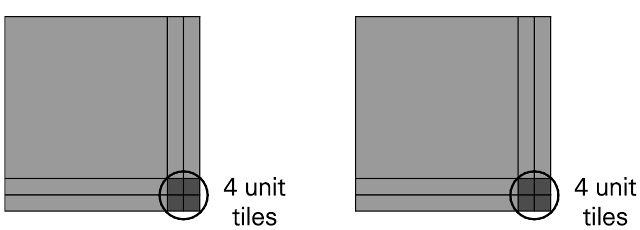
Continue to work with various examples, such as $x^2 + 2x$ and $x^2 + 6x$, to give students an opportunity to describe the pattern that exists. The algebra tile method illustrates the constant term is half the coefficient of the linear term squared.

- When modelling with algebra tiles, it is important for teachers to choose a value of b that is even and relatively small. Before students move from the concrete representation to the pictorial and then to the symbolic method, give students an opportunity to work with expressions where $a > 1$ and $c = 0$. Consider the following example:

  - Ask students to model $2x^2 + 8x$ using algebra tiles and explain why the tiles should be arranged into two equal parts.



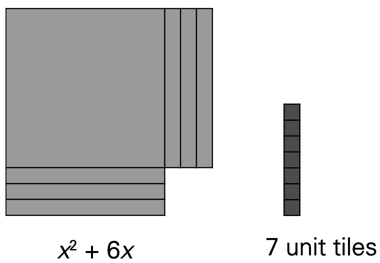
Ask students what tiles must be added to complete the square.



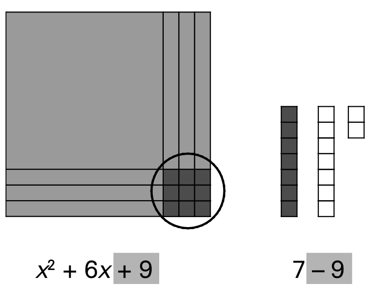
Students will write the expression that represents the perfect square trinomial. They should understand that four unit tiles were added to each diagram creating two squares. The perfect square trinomials can each be written as $x^2 + 4x + 4$ or $(x + 2)^2$. Since there are two squares in the model, the total area is $2(x^2 + 4x + 4)$ or $2(x + 2)^2$.

This visual allows students to gain a better understanding of why the value of b must be divided by *a* before the process of completing the square takes place. That is, $2x^2 + 8x = 2(x^2 + 4x)$.

- Completing the square is an opportunity for students to extend their work with algebra tiles to convert an equation from standard form to vertex form. They will then convert between the two forms algebraically. Use algebra tiles to model the function $y = x^2 + 6x + 7$. Students will continue to split the tiles evenly and add tiles to form a square leaving the constant term alone.

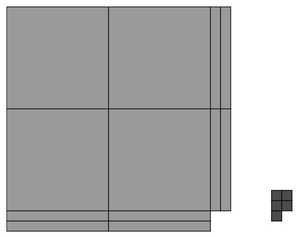
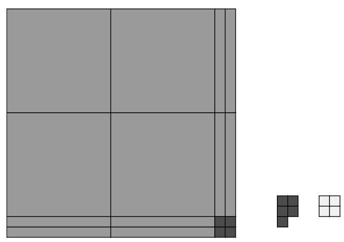
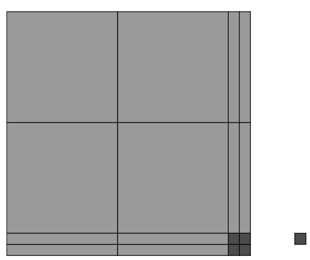


It is important for students to notice that by adding nine extra tiles, the quadratic function has changed. Therefore, students will need to add 9 negative tiles to preserve equality.



The function is $y = (x^2 + 6x \mathbf{+ 9}) + (7 \mathbf{- 9})$. Students can rewrite this as $y = (x + 3)^2 - 2$. It is important for students to recognize when a number of positive tiles are added to form a perfect square, an equivalent of negative tiles must be added to keep the original expression unchanged or preserve equality. The visual representation allows students to observe patterns and then record their work symbolically.

- When students are working with a quadratic with a leading coefficient that is a perfect square such as $y = 4x^2 + 8x + 5$ they may choose to create one large square rather than evenly divide the x^2- and x- tiles into four groups and create four smaller squares.

	$y = 4x^2 + 8x + 5$
	$y = (4x^2 + 8x + 4) - 4 + 5$
	$y = (4x^2 + 8x + 4) + 1$ $y = 4(x^2 + 2x + 1) + 1$ $y = 4(x + 1)^2 + 1$

- You may wish to photograph the group as they participate in an algebra tile task. After a few classes, give students their photograph and ask them to describe what they were doing in the picture. They should write about the task under the photograph, describing what they were doing and what they learned as a result.

- Use technology to demonstrate that a quadratic function in the form $y = ax^2 + bx + c$ will represent the same function when written in the form $y = a(x - p)^2 + q$.

- Use technology (i.e., Tracker software) to write a quadratic equation that models the trajectory of a projectile. An example of this task can be viewed on *YouTube* (i.e., Dan Meyer on Real-World Math, throwing a basketball in a hoop).

- Common errors occur when converting a quadratic function from standard form to vertex form. A quadratic function where $a \neq 1$, for example, sometimes causes difficulty for students. Using algetiles to model completing the square will reduce situations where students make common errors. It is valuable to consider the following correct example and discuss with students the possible errors that could have occurred.

  (a) $y = -3x^2 + 18x - 23$
  (b) $y = -3(x^2 - 6x) - 23$
  (c) $y = -3(x^2 - 6x + 9) - 23 + 27$
  (d) $y = -3(x - 3)^2 + 4$

  – The common factor (–3) is not factored out from both the quadratic and linear terms.
  – There is an incorrect sign on the linear term when a negative leading coefficient is factored out.
  – The constant term inside the parentheses is doubled instead of squared.
  – When a perfect square is created, the constant term inside the parentheses is not multiplied by the common factor to produce the compensated term.
  – The perfect square trinomial is incorrectly factored.

- A few possible activities that reinforce completing the square are described below.

  – Ask students to work in pairs for this task. Give each pair of students a quadratic equation in standard form and ask them to rewrite the equation in vertex form. Ask one student to write the first line of the solution and then pass it to the second student. The second student will verify the workings to determine if an error is present. If there is an error present, the student will correct it and then write the second line of the solution and pass it along to their partner. This process continues until the solution is complete.

  – In groups of two, have students play the Domino Game. Provide students with 10 domino cards. One side of the card will contain a quadratic function in standard form, while the other side will contain a quadratic function in vertex form. Have students lay out the dominos so that the standard form on one card will match with the correct vertex form on another. Students will form a complete loop, with the first card matching the last card. Some sample cards are shown below.

$y = (x + 5)^2 - 3$	$y = 2x^2 - 12x + 23$
$y = 2(x - 3)^2 + 5$	$y = -x^2 - 2x + 2$
$y = -(x + 1)^2 + 3$	$y = x^2 + 10x + 22$

  – In groups of two, ask students to move around the classroom to various stations where solutions have been posted outlining the process of completing the square. At each station the solution could contain one or more errors that the group has to identify and then produce correct solutions for each of the problems.

SUGGESTED MODELS AND MANIPULATIVES

- algebra tiles

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- completing the square
- maximum value
- minimum value
- vertex
- *x*-coordinate
- *y*-coordinate

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 3.2–3.3, pp. 163–197
  - Section 4.1, pp. 206–217
  - Sections 4.3–4.4, pp. 234–258

Notes

SCO RF05 Students will be expected to solve problems that involve quadratic equations. [C, CN, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF05.01 Explain, using examples, the relationship among the roots of a quadratic equation, the zeros of the corresponding quadratic function, and the *x*-intercepts of the graph of the quadratic function.

RF05.02 Derive the quadratic formula, using deductive reasoning.

RF05.03 Solve a quadratic equation of the form $ax^2 + bx + c = 0$ by using strategies such as
- determining square roots
- factoring
- completing the square
- applying the quadratic formula
- graphing its corresponding function

RF05.04 Select a method for solving a quadratic equation, justify the choice, and verify the solution.

RF05.05 Explain, using examples, how the discriminant may be used to determine whether a quadratic equation has two, one, or no real roots, and relate the number of zeros to the graph of the corresponding quadratic function.

RF05.06 Identify and correct errors in a solution to a quadratic equation.

RF05.07 Solve a problem by
- analyzing a quadratic equation
- determining and analyzing a quadratic equation

Scope and Sequence

Mathematics 10 / Mathematics 11	**Pre-calculus 11**	**Pre-calculus 12**
AN01 Students will be expected to demonstrate an understanding of factors of whole numbers by determining the prime factors, greatest common factor, least common multiple, square root, and cube root. (M10)* **AN05** Students will be expected to demonstrate an understanding of common factors and trinomial factoring, concretely, pictorially, and symbolically. (M10) **RF01** Students will be expected to interpret and explain the relationships among data, graphs, and situations. (M10)	**RF05** Students will be expected to solve problems that involve quadratic equations.	**T05** Students will be expected to solve, algebraically and graphically, first and second degree trigonometric equations with the domain expressed in degrees and radians. **RF10** Students will be expected to solve problems that involve exponential and logarithmic equations.

Mathematics 10 / Mathematics 11 (continued)	Pre-calculus 11	Pre-calculus 12
RF05 Students will be expected to Determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. (M10) **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF01** Students will be expected to model and solve problems that involve systems of linear inequalities in two variables. (M11)** **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)		

* M10—Mathematics 10
** M11—Mathematics 11

Background

In this unit, students will extend their factoring skills and use a variety of strategies to determine the roots of quadratic equations. In Mathematics 11 (11RF02), students learned about the connection that the *x*-intercepts of the graph or the zeros of the quadratic function correspond to the solutions, or roots, of the quadratic equation.

In Mathematics 11, students have learned what it means to solve a quadratic equation graphically as well as the limitations of this method. Students have also worked with quadratics algebraically and decided when it is best to factor, or to use the quadratic formula (11RF02). In this course they will add completing the square to their methods of solution for a quadratic equation.

In Mathematics 11, students have worked with the graph of a quadratic function and the points where a parabola crosses the *x*-axis. They are aware a quadratic function can have zero, one, or two *x*-intercepts. When solving a quadratic equation of the form $ax^2 + bx + c = 0$, students can graph the corresponding quadratic function and determine the *x*-intercepts. They can use a table of values or graphing technology to make the connection between the *x*-intercepts of the graph and the roots of the quadratic equation.

For this outcome, students will explore the relationship between roots, *x*-intercepts, and zeros in more detail. They will derive the quadratic formula using deductive reasoning and then identify and use the value of the **discriminant** ($b^2 - 4ac$) to determine the type of roots and how this relates to the graph of the function.

- If $b^2 - 4ac > 0$, there are two distinct real roots, and two x-intercepts.
- If $b^2 - 4ac = 0$, there are two real equal roots, and one x-intercept.
- If $b^2 - 4ac < 0$, the roots are imaginary, and there are no x-intercepts.

Having added completing the square and using the discriminant to their toolkit, students will practice solving quadratic equations in a variety of ways, as appropriate to the situation and problem to be solved.

Mathematics 11 (RF02) students learned to distinguish between the terms **roots**, **zeros,** and ***x*-intercepts**, and to use the correct term in a given situation. The x-intercepts of the graph or the zeros of the quadratic function correspond to the roots of the quadratic equation. Students were asked to find the roots of the equation $x^2 - 7x + 12 = 0$, find the zeros of $f(x) = x^2 - 7x + 12$, or determine the x-intercepts of the graph of $y = x^2 - 7x + 12$. In each case they were solving $x^2 - 7x + 12 = 0$ and arriving at the solution $x = 3$ or $x = 4$.

In Mathematics 11 (RF02), when students are exposed to quadratic equations that could be factored, they used the quadratic formula to determine a solution. They now have the option to complete the square or apply the quadratic formula in order to find the exact solution(s).

Students used the quadratic formula in Mathematics 11 without deriving it. At this point it is important for students to understand how the quadratic formula is derived. They should use a numerical example before moving to the general form $ax^2 + bx + c = 0$. Ask them to complete the square using an example similar to $3x^2 - 7x + 1 = 0$. Assist students as they follow the same procedure to derive the quadratic formula for $ax^2 + bx + c = 0$.

$$\begin{aligned}
&3x^2 - 7x + 1 = 0\\
&3(x^2 - \frac{7}{3}x) + 1 = 0\\
&3(x^2 - \frac{7}{3}x + \frac{49}{36}) - \frac{49}{12} + 1 = 0\\
&3(x - \frac{7}{6})^2 = \frac{49}{12} - 1\\
&3(x - \frac{7}{6})^2 = \frac{49}{12} - \frac{12}{12}\\
&3(x - \frac{7}{6})^2 = \frac{37}{12}\\
&(x - \frac{7}{6})^2 = \frac{37}{36}\\
&x - \frac{7}{6} = \pm\frac{\sqrt{37}}{6}\\
&x = \frac{7}{6} \pm \frac{\sqrt{37}}{6}\\
&x = \frac{7 \pm \sqrt{37}}{6}
\end{aligned}$$

$$\begin{aligned}
&ax^2 + bx + c = 0\\
&a(x^2 + \frac{b}{a}x) + c = 0\\
&a(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}) - \frac{ab^2}{4a^2} + c = 0\\
&a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + c = 0\\
&a(x + \frac{b}{2a})^2 - \frac{b^2}{4a} + \frac{4ac}{4a} = 0\\
&a(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a}\\
&(x + \frac{b}{2a})^2 = \frac{b^2 - 4ac}{4a}\\
&x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{4a^2}\\
&x + \frac{b}{2a} = \pm\frac{\sqrt{b^2 - 4ac}}{2a}\\
&x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}\\
&x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\end{aligned}$$

Radical and rational expressions are covered in detail in this course. Depending on what order the topics are taught, students may need assistance when adding rational expressions $\left(-\frac{b^2}{4a}+c\right)$ and simplifying variable roots $\sqrt{4a^2}$.

It is beneficial to have students analyze solutions that contain errors. Students should be provided with worked solutions of quadratic equations that may or may not contain errors. If errors are present, students should identify the error and provide the correct solution including why/how the error occurred. This reinforces the importance of recording solution steps rather than only giving a final answer.

As students work through the various strategies for solving quadratic equations, they should learn that sometimes one method is more efficient than another. The method students choose to solve a quadratic equation will depend on the form of the equation (i.e., vertex form, standard form, or factored form).

Students have solved quadratic equations graphically and algebraically. Regardless of their strategy to solve the equation, they should understand that their algebraic solution is the same as the *x*-intercepts of the graph.

When students use the quadratic formula to find roots of a quadratic equation, they will be exposed to trying to find the square roots of non-perfect squares and negative numbers. It is important for students to distinguish between exact solutions and approximate solutions.

Although it is not the intention of this course to introduce students to the imaginary number system, they need to be aware that there is no real solution to the square root of a negative number. This will lead to a discussion regarding the conditions that are necessary for the quadratic formula to result in two real roots, one real root, and no real solutions (two non-real solutions).

Contextual problems will also be solved by modelling a situation with a quadratic equation.

Quadratic equations can be used to model a variety of situations such as projectile motion and geometry-based word problems. Students should be exposed to examples that require them to model the problem using a quadratic equation, solve the equation, and interpret the solution.
Consider the following example:

- A rectangular lawn measuring 8 m × 4 m is surrounded by a flower bed of uniform width. The combined area of the lawn and flower bed is 165 m^2. What is the width of the flower bed?

It is important for students to understand that the context of the problem dictates **inadmissible roots**. Discuss with students different scenarios that produce inadmissible roots. For example, time, height, and length, would not make sense if they have a negative numerical value. However, temperature could be both negative and positive.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

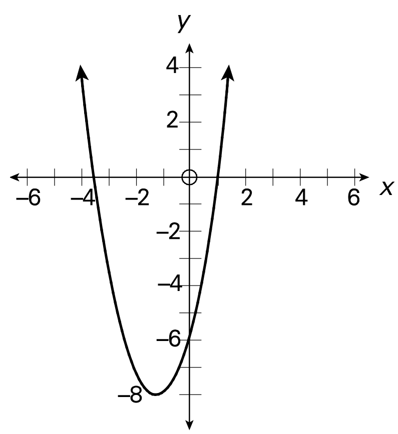
Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- For the function $f(x) = 2x^2 + 5x - 7$
  (a) Find the zeros of $f(x) = 2x^2 + 5x - 7$.
  (b) Identify the x-intercepts of the graph.



  (c) Find the roots of $2x^2 + 5x - 7 = 0$.
  (d) What do you notice about the answers to the above tasks?

- Solve the equation $6x^2 + 7x - 3 = 0$.

- Explain how it is possible to determine how many x-intercepts a quadratic function has without graphing the function.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Identify the most efficient strategy (i.e., determining square roots, factoring, completing the square, applying the quadratic formula, or graphing its corresponding function) when solving each equation. Justify your choice.
  (a) $(x + 2)^2 - 1 = 15$
  (b) $49x^2 - 64 = 0$
  (c) $3x^2 - 11x + 6 = 0$
  (d) $x^2 - 10x - 24 = 0$
  (e) $3x^2 + 8x + 7 = 0$

- Given the quadratic equation $ax^2 + bx + c = 0$, what relationship must be true for the coefficients *a*, *b*, and *c* so that the equation has
  (a) two distinct real roots
  (b) two equal real roots
  (c) no real roots

- Sheena, David, and Darius are students in a group. They are given the equation $A = x^2 + 3x - 110$ where *A* represents the area of a field and *x* represents the width in metres. The students were asked to find the width if the area was 100 m^2. Each student decided to solve the equation using their own preferred method. Here are their solutions:

  Sheena
  $x^2 + 3x - 110 = 100$
  $x^2 + 3x - 210 = 0$
  $x = \frac{3 \pm \sqrt{9 - (4)(1)(-210)}}{2}$
  $x = \frac{3 \pm \sqrt{-831}}{2}$
  $x = \frac{3 \pm 28.827}{2}$
  $x = 15.9$ or $x = -12.9$
  width is 15.9 m

  David
  $x^2 + 3x - 110 = 100$
  $x^2 + 3x - 10 = 0$
  $(x+5)(x-3) = 0$
  $x = -5$ or $x = 3$
  width is 3 m

  Darius
  $x^2 + 3x - 110 = 0$
  $x = \frac{-3 \pm \sqrt{9 - (4)(1)(-110)}}{2}$
  $x = \frac{-3 \pm \sqrt{9 + 440}}{2}$
  $x = \frac{-3 \pm \sqrt{449}}{2}$
  $x = \frac{-3 \pm 21.2}{2}$
  $x = 9.1$ or $x = -12.1$
  width is 9.1 m

  Identify and explain any errors in the students' work. Write the correct solution.

- When Chantal was asked to describe the roots of the equation $14x^2 - 5x = -5$, she rearranged the equation so that it would equal zero, then used the quadratic formula to find the roots. Her work is shown below. Edward said that she didn't have to do all that, and he then showed the class his work. Are they are both correct? Which method do you prefer? Explain your reasoning.

Chantal	**Edward**
$x=\dfrac{5\pm\sqrt{25-280}}{28}$	b^2-4ac
$x=\dfrac{5\pm\sqrt{-255}}{28}$	$=25-280$
no real roots	$=-255$
	no real roots

- For what values of t does $x^2 + tx + t + 3 = 0$ have one real root?

- Show that if the quadratic equation $px^2 + (2p + 1)x + p = 0$ has two real unequal roots, then $4p + 1 > 0$.

- Assume a, b, and c are real numbers. How many times would $y=ax^2+bx+c$, intersect the x-axis if the discriminant of $ax^2+bx+c=0$ is
  (a) positive
  (b) zero
  (c) negative

- Create three quadratic equations. One that has two distinct roots, one that has two equal roots, and one that has no real roots.

- A ball is thrown from a building at an initial height of 11 metres and reaches a maximum height of 36 metres, 5 metres from the building.
  (a) Write a quadratic equation that models this situation where the x represents the number of metres from the building.
  (b) Three targets are placed at different locations on the ground. One is at (10, 0), another at (11, 0) and a final target is placed at (12, 0). Which target does the ball hit? Explain how you arrived at your answer.

- Find two consecutive whole numbers such that the sum of their squares is 265.

- A diver's path when diving off a platform is given by $d = -5t^2 + 10t + 20$, where d is the distance above the water (in feet) and t is the time from the beginning of the dive (in seconds).
  (a) How high is the diving platform?
  (b) When is the diver 25 feet above the water?
  (c) When does the diver enter the water?

- Choose a quadratic word problem from the class notes or group workstations and use it as a guide to create your own word problem (use a real-life situation that you are interested in). Remember to include your solutions on a separate sheet.

- For each of the following, explain the relationship between the roots of the equation, the zeros of the corresponding function, and the x-intercepts of that function.
  (a) $f(x) = x^2 + 11x + 30$
  (b) $f(x) = x^2 - 12x + 36$
  (c) $f(x) = 4x^2 + 2x + 3$

- Show how the quadratic formula can be derived based on the idea of completing the squares.

- Sketch graphs for quadratic functions for which the discriminant is less than zero, equal to zero, and greater than zero.

- Write quadratic equations for each of the following:
  (a) The discriminant is less than zero.
  (b) The discriminant is equal to zero.
  (c) The discriminant is greater than zero.

- Solve the following using the most efficient method.
  (a) $x^2 - 449 = 0$
  (b) $2x^2 + x - 6 = 0$
  (c) $6x^2 - 30x - 58.5 = 0$
  (d) $4x^2 - 18x = 10$
  (e) $x^2 = 5x + 20$

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Provide students with a variety of equations to evaluate and, working in groups of two or three, have each group determine which methods are most appropriate and/or efficient for solving the quadratic equation. Have them present to each other and explain their reasoning.

- Allow students a significant amount of time to explore the most efficient methods to use to solve various quadratic equations in the form $ax^2 + bx + c = 0$. Let students generate some general rules to guide them in approaching new questions. For example,
  - graphing may be best if graphing calculators are available
  - factoring either algebraically or with algebra tiles may be appropriate if numbers are simple squares
  - completing the square may be used if factoring is difficult but fractions or decimals are not involved
  - the quadratic formula may need to be used if numbers are fractions or decimals

- Some common errors occur when students are simplifying the quadratic formula. These include
  - applying the quadratic formula without ensuring the equation is written in standard form
  - using $x = -b \pm \dfrac{\sqrt{b^2 - 4ac}}{2a}$ rather than the correct form of the quadratic formula.
  - incorrectly producing two possible common errors if the b value is negative.
    - > If $b = -2$ then $-b = -(-2) = -2$
    - > If $b = -2$ then $b^2 = -2^2 = -4$
  - incorrectly simplifying when applying the quadratic formula
    - > $\dfrac{8 \pm \sqrt{5}}{2} = 4 \pm \sqrt{5}$
    - > $\dfrac{2 \pm 4\sqrt{5}}{2} = \pm 2\sqrt{5}$
  - not recognizing that the $\pm$ results in two solutions (Suggest that students work through the solutions separately, showing calculations for both the positive solution and the negative solution.)

- When modelling situations for students, emphasize that restrictions sometimes need to be placed on the independent variable of the function. If a solution does not lie in the restricted domain, then it is not a solution to the problem. The following is an example with a **restricted domain**.
  - A baseball is thrown from an initial height of 3 m and reaches a maximum height of 8 m, two seconds after it is thrown. At what time does the ball hit the ground?

- In the above example, the quadratic equation only models the path of the ball from the time it leaves the throwers' hand to the time it makes first contact with the ground. This quadratic equation yields two possible solutions, one of which is negative. This implies that it occurred before the ball was thrown. The restriction on the domain causes the negative solution to be inadmissible since time cannot be negative and only the positive solution is accepted.

- Ask students to solve each of the following using the quadratic formula.
  (a) $2x^2 + 10x + 3 = 0$
  (b) $x^2 + 6x + 9 = 0$
  (c) $2x^2 + 3x + 5 = 0$

  As teachers observe students' work, use the following prompts to promote discussion:
  - What is the value under the square root in each equation? What does this tell you about the roots of the equation?
  - What connection can be made between the value of $b^2 - 4ac$ and the number of real roots an equation has?
  - What values of $b^2 - 4ac$ could lead to approximate answers?

- One possible task that reinforces solving a quadratic is described below.

- Two sets of different coloured cards are required for this task. One set will contain quadratic equations and the other set will have their corresponding solutions. Ask students to lay out the cards and match the equation card with its corresponding solution card.

$x^2 - 2x - 24 = 0$	$x = \frac{5}{2}, x = 2$	$2x^2 + 5x + 3 = 0$	$x = -\frac{3}{2}; x = -1$
$2x^2 - 5x + 3 = 0$	$2x^2 + 7x + 5 = 0$	$x = -\frac{3}{2}; x = 1$	$x = 4; x = -6$
$x^2 + 2x - 24 = 0$	$2x^2 - 9x + 10 = 0$	$2x^2 - x - 3 = 0$	$x = -4; x = 6$
$2x^2 + x - 3 = 0$	$x = -\frac{5}{2}; x = -1$	$x = \frac{3}{2}; x = 1$	$x = -\frac{3}{2}; x = -1$

Suggested Models and Manipulatives

- grid paper
- index cards

Mathematical Vocabulary

Students need to be comfortable using the following vocabulary.

- discriminant
- extraneous root
- quadratic equation
- quadratic formula
- roots of a function
- zeros of a function

Resources/Notes

Internet

- *Math Is Fun*, “Quadratic Equation Solver” (MathIsFun.com 2014)
  www.mathisfun.com/quadratic-equation-solver.html
  An interactive site consisting of a quadratic solver and its graphical representation.

- www.k12pl.nl.ca/curr/10-12/math/math2201/classroomclips/quadeq.html
  The Quadratic Equation clip demonstrates students taking turns solving a quadratic equation using the quadratic formula.

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 4.1–4.4, pp. 206–257

Notes

SCO RF06 Students will be expected to solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables. [CN, PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

(*It is intended that the quadratic equations be limited to those that correspond to quadratic functions.*)

RF06.01 Model a situation, using a system of linear-quadratic or quadratic-quadratic equations.

RF06.02 Relate a system of linear-quadratic or quadratic-quadratic equations to the context of a given problem.

RF06.03 Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations graphically, with technology.

RF06.04 Determine and verify the solution of a system of linear-quadratic or quadratic-quadratic equations algebraically.

RF06.05 Explain the meaning of the points of intersection of a system of linear-quadratic or quadratic-quadratic equations.

RF06.06 Explain, using examples, why a system of linear-quadratic or quadratic-quadratic equations may have zero, one, two, or an infinite number of solutions.

RF06.07 Solve a problem that involves a system of linear-quadratic or quadratic-quadratic equations, and explain the strategy used.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
RF09 Students will be expected to represent a linear function, using function notation. (M10)* **RF10** Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically. (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**RF06** Students will be expected to solve, algebraically and graphically, problems that involve systems of linear-quadratic and quadratic-quadratic equations in two variables.	**RF14** Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials).

* M10—Mathematics 10

** M11—Mathematics 11

Background

In Mathematics 10 students solved systems of linear equations in two variables, graphically and algebraically (RF09). They created a linear system to model a situation, as well as wrote a description of a situation that might be modelled by a given linear system (RF04). They solved linear systems graphically, with and without technology, and progressed to solving linear systems symbolically using substitution and elimination.

In this course, this learning will be extended to solving systems of **linear-quadratic** equations and **quadratic-quadratic equations**. This involves finding the intersection points between a line and a parabola (linear-quadratic) or between two parabolas (quadratic-quadratic).

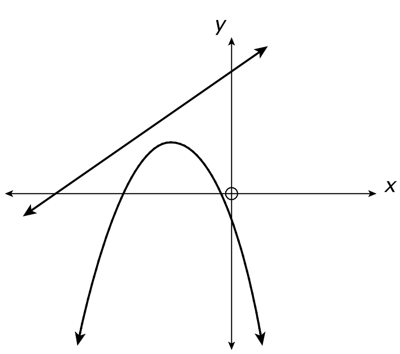
The statement made prior to the performance indicators for this outcome “It is intended that the quadratic equations be limited to those that correspond to quadratic functions, ” refers to the fact that only quadratics equations that can be written in the form $y = ax^2 + bx + c$, $a \neq 0$ will be considered as part of this outcome.

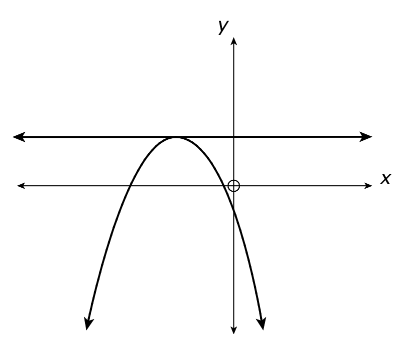
Quadratic relations can be described by the **general equation** $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$. Only those where B = 0, C = 0, and E = 0 describe functions.

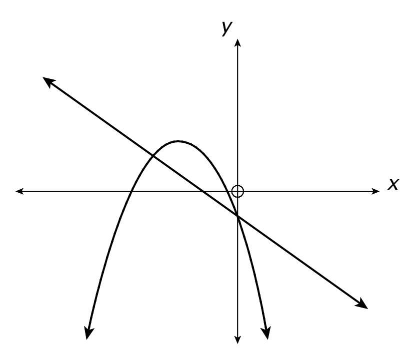
For example, quadratic relations such as $x^2 + y^2 = 1$ (the equation of a circle), $x^2 - y^2 = 1$ (the equation of a hyperbola), and $x = y^2$ (the equation of a parabola opening to the right) are not a part of this course. They are not functions.

In Mathematics 10, students created graphs of linear functions (RF06). They also investigated the meaning of the point of intersection of a system of linear equations (RF09). Students discovered that the intersection points of the graphs represented the solution to the system of equations.

When students sketch a line and a parabola on the same axis and examine their intersection points, they should recognize there are three possible situations for a linear-quadratic system. It can have zero solutions, one solution, or two solutions.







It is not expected that the concepts of a tangent or a secant line be discussed as a part of this course. However, if a line intersects the parabola more than once, the line is referred to as a **secant line** and if the line intersects the parabola at exactly one point, the line is called a **tangent line**. Exposure to an understanding of this terminology will be an asset to future studies of calculus.

When students draw two parabolas that are functions and examine how many solutions are possible for a quadratic-quadratic system, they will recognize that there can be zero, one, two or an infinite number of solutions (i.e., coincident).

Students will be working with the graphs of linear-quadratic and quadratic-quadratic systems to find their solutions graphically. Students will need to revisit the methods used for graphing linear and quadratic relations. In Mathematics 10, students were exposed to graphing linear functions using the slope-intercept method, slope-point method, and using the x-intercept and y-intercept method. In Mathematics 11, students were exposed to graphing quadratic functions using the vertex and y-intercept. Earlier in this course, students were exposed to the use of transformations when graphing a quadratic.

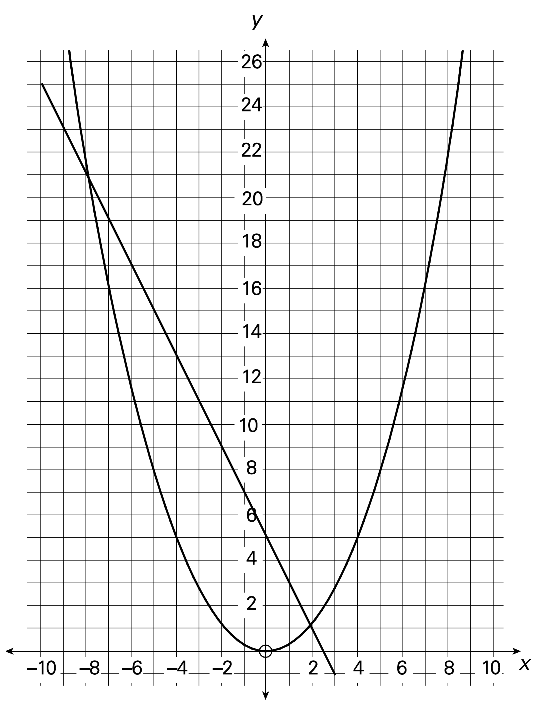
Identifying the form of the equation will help students decide which method they should choose when graphing the linear or quadratic function. Graphing technology can also be used to solve a system of equations. Students can be exposed to, but not limited to, a graphing calculator or graphing applications.

When students identify the points of intersection, remind them to verify the solution for both equations. There are limitations to solving a system by graphing. Non-integral intersection points are possible where students will have to estimate the coordinates or the solutions may be off the calculator screen when graphed, making an adjustment of window necessary. In such cases, an algebraic method of solving these systems is often more efficient.

Through their work with solving systems graphically, students should understand that algebra provides a more efficient means of finding the points of intersection. In Mathematics 10, students were exposed to algebraic methods of substitution and elimination to solve a linear system. They will apply these methods to systems involving quadratic equations, including ones with rational coefficients. Provide students an opportunity to decide which algebraic method is more efficient when solving a system by focusing on the coefficients of like variables. If necessary, ask students to rearrange the equations so that like variables appear in the same position in both equations.

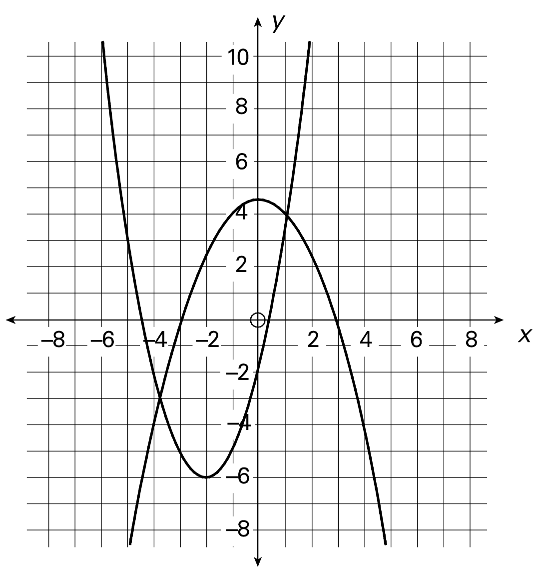
Algebraically, linear-quadratic systems can be solved most efficiently by substitution. For example,

<table>
<tr><td colspan="2">$2x+y=5$ and $x^2-3y+3=4$</td></tr>
<tr><td>Rearranging $2x+y=5$ to obtain $y=5-2x$
We use substitution for y.
$x^2-3(5-2x)+3=4$
$x^2-15+6x+3=4$
$x^2+6x-12=4$
$x^2+6x-16=0$
$(x+8)(x-2)=0$
$x+8=0 \quad x-2=0$
$x=-8 \qquad x=2$</td><td>Then substitute the value(s) of x to obtain the y-coordinates.
$2x+y=5$
$2(-8)+y=5$
$-16+y=5$
$y=21 \quad \therefore$ point $(-8,21)$

$2x+y=5$
$2(2)+y=5$
$4+y=5$
$y=1 \quad \therefore$ point $(2,1)$</td></tr>
<tr><td colspan="2">Verifying the intersection point(s) graphically.
</td></tr>
</table>

Algebraically, quadratic-quadratic systems can most efficiently be solved by substitution and sometimes by elimination. For example to solve the system $x^2+2y-6=1$ and $x^2-y+4x=2$:

Using substitution	
Rearranging $x^2-y+4x=2$ to obtain $y=x^2+4x-2$ We use substitution for *y* $x^2+2y-6=1$ $x^2+2(x^2+4x-2)-6=1$ $x^2+2x^2+8x-4-6=1$ $3x^2+8x-10=1$ $3x^2+8x-11=0$ $(3x+11)(x-1)=0$ $3x+11=0 \qquad x-1=0$ $x=-\frac{11}{3}\, x=1$	Then substitute the value(s) of *x* to obtain the *y*-coordinates. $x^2-y+4x=2$ $(-\frac{11}{3})^2-y+4(-\frac{11}{3})=2$ $\frac{121}{9}-y-\frac{44}{3}=2$ $-\frac{11}{9}-y=2$ $y=-\frac{29}{9} \quad \therefore \text{point } (-\frac{11}{3},-\frac{29}{9})$ $x^2-y+4x=2$ $(1)^2-y+4(1)=2$ $1-y+4=2$ $5-y=2$ $y=3 \quad \therefore \text{point } (1,3)$
Verifying the intersection point(s) graphically.	



<table>
<tr><td colspan="2">Using elimination</td></tr>
<tr><td>

$x^2+2y-6=1$
$x^2-y+4x=2$

Multiplying the second equation by 2 in order to eliminate the y.
$x^2+2y=7$
$2x^2-2y+8x=4$

Adding the two equations you obtain the quadratic
$3x^2+8x=11$

Solving:
$3x^2+8x-11=0$
$(3x+11)(x-1)=0$
$3x+11=0;\ x-1=0$
$x=-\frac{11}{3};\ x=1$

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Then substitute the value(s) of x to obtain the y-coordinates.
$x^2-y+4x=2$
$(-\frac{11}{3})^2-y+4(-\frac{11}{3})=2$
$\frac{121}{9}-y-\frac{44}{3}=2$
$-\frac{11}{9}-y=2$
$y=-\frac{29}{9}\quad \therefore \text{point } (-\frac{11}{3},-\frac{29}{9})$

$x^2-y+4x=2$
$(1)^2-y+4(1)=2$
$1-y+4=2$
$5-y=2$
$y=3\quad \therefore \text{point } (1,3)$

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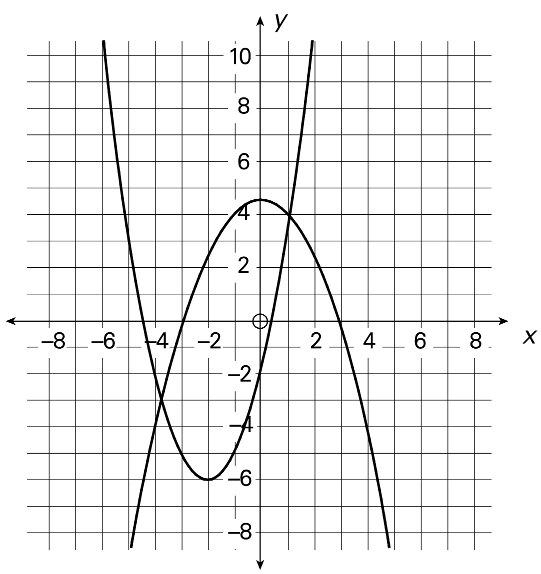
Verifying the intersection point(s) graphically.


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Graphing technology can also be used to solve systems of equations, by using the intersection function to identify the points of intersection. Once found, the solutions should be verified algebraically. The points of intersection of a system of equations will be solutions that will work for both equations.

When students solve the system using substitution or elimination, they will have to use factoring, completing the square (if done prior to this outcome), or the quadratic formula to solve the resulting quadratic equation. After determining the solution to the system, ensure that students verify that the ordered pair satisfies the equations. Discuss with them why it is important to verify the solution in both equations. The check ensures students that they have not graphed incorrectly or made any algebraic mistakes.

Encourage students to predict the number of solutions before they solve the system graphically or algebraically. The discriminant, $b^2 - 4ac$, can be used to help them with their prediction.

Consider the system, $y = 3x + 5$ and $y = 3x^2 - 2x - 4$. Ask students to equate the equations and simplify. Their resulting quadratic equation is $3x^2 - 5x - 9 = 0$. They determine the value of the discriminant to be 133. Students should understand that when the discriminant is greater than zero, there are two *x*-values indicating two different solutions. This means that the linear-quadratic system has two points of intersection. If the discriminant is equal to zero, there is one solution and a negative discriminant means there is no point of intersection. They would then proceed to solve the system and the resulting quadratic equation using methods learned earlier (factoring, quadratic formula, completing the square).

Remind students that a system of quadratic-quadratic equations can also produce an infinite number of solutions. When the functions are coincident (such as $y = x^2 + 6x + 9$ and $y = (x + 3)^2$), students will recognize this results in a system where the left side equals the right hand side (i.e., 0 = 0).

Students will model situations using a system of linear-quadratic and quadratic-quadratic equations connected to a variety of real-life contexts. When they can relate their learning to real-life applications, it has more meaning. Paths of thrown or falling objects can be used to model both quadratic and linear functions. Ensure students understand and define the variables that are being used to represent the unknown quantities. Discuss with students that in order to solve a system and obtain intersection point(s), the number of unknowns must match the number of equations.

Explaining the meaning of a solution in a particular context and verifying the solution is an important part of solving systems. An answer to a system of equations may not necessarily be a possible solution in the context of the situation (i.e., inadmissible root). While solving a system of equations, a student may determine that the solutions are, for example, (–2, 5) and (5, 8). In the context of the problem, the *x*-coordinate may represent a measurement such as time or length. The point (–2, 5), therefore, would have to be rejected as a solution leading to only one possible answer.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

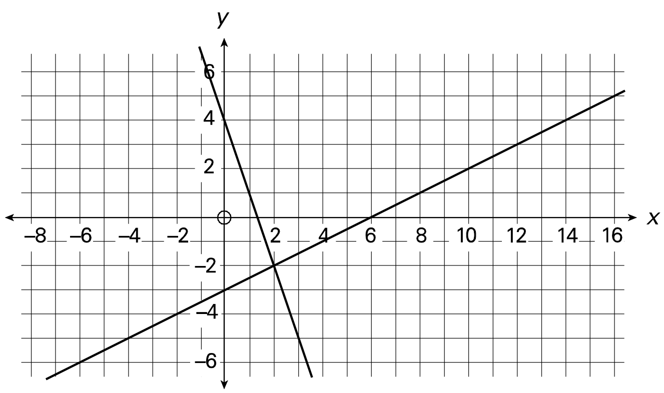
- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Solve the system:
  $y-2x=5$
  $10+3x=4y$
  - Using substitution
  - Using elimination
  - By graphing

- Determine the system of equations illustrated in the graph shown below.



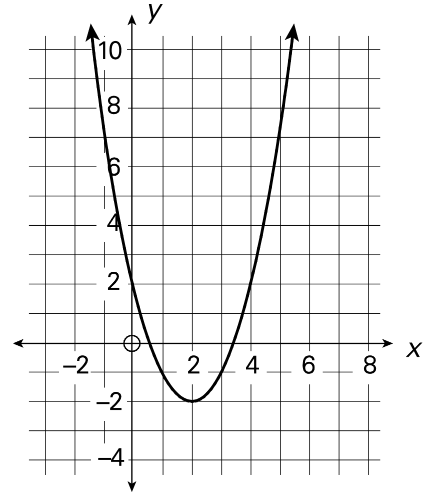
WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Consider the parabola with equation is $y=x^2-2x-3$.
  (a) What is a possible equation for a line such that $y=x^2-2x-3$ and the line form a system with no solution? One solution? Two solutions?
  (b) What is a possible equation of a quadratic that forms a system with $y=x^2-2x-3$ with no solution? One solution? Two solutions? Infinite number of solutions?
- Explain how you can determine, by observation, which of the following systems has no solution and which one has an infinite number of solutions.

(a) $\begin{aligned}-2x^2+3x-y+4=0\\ -4x^2+6x-2y+8=0\end{aligned}$

(b) $\begin{aligned}y-5=0\\ y=-(x+1)^2-3\end{aligned}$



- Do the following lines intersect the parabola shown at zero, one, or two points?

  (a) $y = \frac{1}{2}x + 3$

  (b) $y = \frac{1}{2}x - 4$

  (c) $y = -2x$

  (d) $x - 3 = 0$

- Is it possible for a system of linear-quadratic equations to have an infinite number of solutions? Explain your reasoning.

- Sam solved the system $\begin{matrix} 2x - y = 9 \\ y = x^2 - 4x \end{matrix}$.

  His solution was (3, –3). Verify whether his solution is correct and explain how Sam's results can be illustrated on a graph.

- Ayla was asked to solve the system $\begin{matrix} x^2 - x + y = -2 \\ 2x^2 - 4x + 3y = 0 \end{matrix}$. The beginning of his solution is shown below.

  Finish the solution and verify your answer. Identify and correct any error(s) you or Ayla may have made.

  Ayla's Solution:
  Multiply the first equation by –3 and add the second equation.

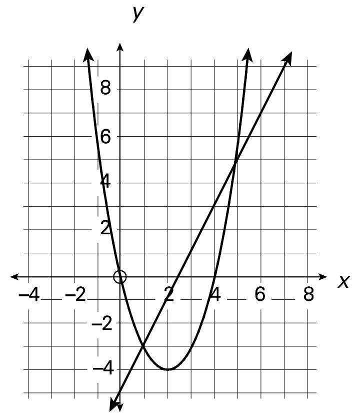
$$x^2 - x + y = -2 \xrightarrow{x(-3)} -3x^2 + 3x - 3y = -2$$

$$2x^2 - 4x + 3y = 0 \longrightarrow \begin{array}{r} 2x^2 - 4x + 3y = 0 \\ \hline -x^2 - x \quad = -2 \end{array}$$

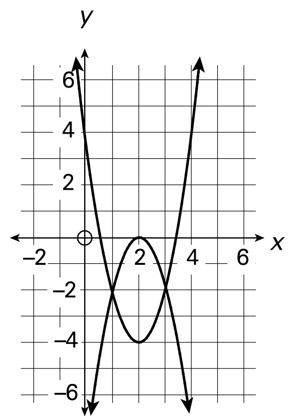
  Now, solve $-x^2 - x = -2$

- Solve the system of equations represented below. Verify that the solutions obtained algebraically match those found graphically.

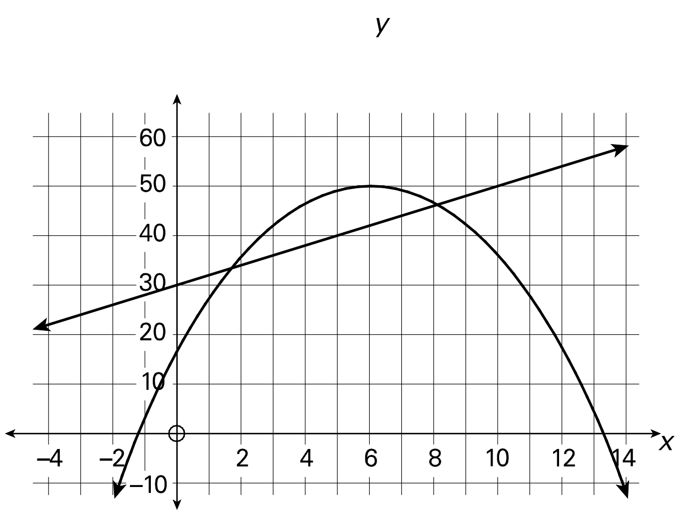
(a) $y + 5 = 2x$
$y = (x-2)^2 - 4$



(b) $8x - y = 2(x^2 + 4)$
$y = 2(x-2)^2 - 4$



- The price C, in dollars per share, of a high-tech stock has fluctuated over a twelve-year period and is represented by the parabola shown. The price C, in dollars per share, of a second high-tech stock has shown a steady increase during the same time period.



(a) Determine the system of equations that models the price over time.
(b) Solve the system.
(c) Explain the meaning of the intersection points of this system of equations.

- A rectangular field has a perimeter of 500 m and an area of 14 400 m^2. Find the length of the sides.

- Write a system of equations to represent two numbers that differ by 4 and whose squares have a sum of 136?

- A right triangle has a hypotenuse 10 cm long. If the perimeter is 22 cm, find the lengths of the other two sides.

- A sky diver jumped from a tower and fell freely for several seconds before releasing her parachute. Her height, *h*, in metres, above the ground at any time is given by $h = -4.9t^2 + 5000$ before she released her parachute, and $h = -4t + 4000$ after she released her parachute. If t represents time in seconds, how long after jumping did she release her parachute? How high was she above the ground at that time?

- State the number of possible solutions to each of the following systems. Include graphs to support your answers.
  (a) A quadratic equation and a horizontal line.
  (b) Two quadratics with positive leading coefficients (values of *a* for $y = ax^2 + bx + c$).
  (c) A quadratic and a line with negative slope.
  (d) A quadratic and a vertical line.

- Solve the following system of equations.
  $5x^2 + 3y = -3 - x^2$
  $x^2 - x = -4 - 2y$

- From the base of a hill with a constant slope, Steve hits a golf ball as hard as he can up the hill. The system of equations below could model this situation.

  $h(d) = -0.18d^2 + 3.6d$
  $h(d) = 12d$

  (a) Solve the system.
  (b) If *d* represents horizontal distance, *h* represents vertical height, interpret the points of intersection in the context of the problem.

  (**Answer:** One point of intersection represents the location on the hill where the ball lands, the other is the starting point of the ball.)

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- A quick review of solving systems of linear equations using substitution, elimination, and graphing should be used to introduce this unit.

- Contextual situations such as the following can help students visualize the meaning of the solutions. Students may find it helpful to use technology to graph these situations as well as solve them algebraically.

  - **For a linear-quadratic system with zero solutions**
    As a science experiment, Devon releases a balloon from the top of a 64-foot high building at the same time that Mathias shoots an arrow from the ground. The equation describing the height (y) of the arrow in feet is $y = 1 + 20x - 16x^2$ where x is seconds. The equation describing the height of the balloon in feet is $y = 6x + 64$. After how many seconds will the balloon and the arrow pass each other?

  - **For a linear-quadratic system with one solutions**
    As a science experiment Sanjia drops a toy from the top of a 64-foot high building at the same time that Ethan releases a Happy Birthday balloon from the ground. The equation describing the height (y) above ground of the toy in feet is $y = 64 - 2x^2$ where x is seconds. The equation describing the elevation of the balloon in feet is $y = 6x + 8$. After how many seconds will the balloon and quarter pass each other?

  - **For a linear-quadratic system with two solutions**
    A rocket is launched from the ground and follows a parabolic path represented by the equation $y = -x^2 + 10x$. At the same horizontal location, a flare is launched from a height of 10 feet and follows a straight path represented by the equation $y = -x + 10$. Graph the equations that represent the paths of the rocket and the flare, and find the coordinates of the point or points where the paths intersect.

- **For a quadratic-quadratic system with a single solution**
  A ball is thrown into the air so that its vertical height, in metres, in terms of the horizontal distance, in metres, it is from the edge of the water fountain is given by $h(x) = -4.9x^2 + 20x + 1$.
  A large water fountain has a stream of water forming a parabolic path so that the height, in metres, of the stream of water is given by $h(x) = -4.9x^2 + 22x$ in terms of the horizontal distance, in metres, from the edge of the fountain. Determine where the ball passes through the stream of water.

- **For a quadratic-quadratic system with no solution or an infinite number of solutions**
  (a) The fine for speeding in one province is determined by adding $40 court costs to the square of the number of kmh the individual was exceeding the speed limit. State an equation that could be used to calculate the fine for speeding in this province.
  (b) In a second province the fine is determined by adding $80 to double the square of the number of kmh the individual was exceeding the speed limit. State an equation that could be used to calculate the fine for speeding in this province.
  (c) When will the fines in the two provinces be the same?
  (d) In the second province, a first-time offender is charged half the regular fine. State an equation that could be used to calculate the fine for a first-time offender.
  (e) When will the fines in the two provinces be the same?

- **For a quadratic-quadratic system with two solutions**
  A rocket is launched into the air so that its vertical height, in feet, in terms of the horizontal distance, in feet, it is from the edge of the water fountain is given by $h(x) = 21.5x^2 + 3x + 0.5$.
  A large water fountain has a stream of water forming a parabolic path so that the height, in feet, of the stream of water is given by $h(x) = -16x^2 + 41x$ in terms of the horizontal distance, in feet, from the edge of the fountain. Determine where the rocket passes through the stream of water.

- Students should explore the various possibilities for solution of a linear-quadratic and a quadratic-quadratic system of equations. Asking students to use a graphing utility to find a system that has a certain number of solutions (zero, one, or two) and then verify this using algebra is a task that reinforces this visual connection to the algebraic solution.

- It is intended that students will use graphing technology to verify their algebraic solutions to the problems in this unit.

- **Task for concept reinforcement**
  Cards are distributed amongst the students. One third of the cards contain the coordinates of the points of intersection of two graphs. Another third of the cards contain a system of linear-quadratic or quadratic-quadratic equations written in slope *y*-intercept and vertex form. The final third contains the corresponding graphs for the system of equations. Students move around the classroom attempting to form a group of three by matching the cards containing the corresponding systems, graphs, and points of intersections.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- coincident
- discriminant
- elimination
- inadmissible root
- intersection point
- linear-quadratic system
- quadratic-quadratic system
- solution
- substitution
- system of equations

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 8.1–8.2, pp. 424–456

Notes

SCO RF07 Students will be expected to solve problems that involve linear and quadratic inequalities in two variables. [C, PS, T, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF07.01 Explain, using examples, how test points can be used to determine the solution region that satisfies an inequality.
RF07.02 Explain, using examples, when a solid or broken line should be used in the solution for an inequality.
RF07.03 Sketch, with or without technology, the graph of a linear or quadratic inequality.
RF07.04 Solve a problem that involves a linear or quadratic inequality.

Scope and Sequence

Mathematics 10 / Mathematics 11	**Pre-calculus 11**	**Pre-calculus 12**
RF06 Students will be expected to **r**elate linear relations to their graphs expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + By + C = 0$) ▪ slope-point form ($y - y_1) = m(x - x_1$) (M10)* **RF09** Students will be expected to represent a linear function, using function notation. (M10) **RF10** Students will be expected to solve problems that involve systems of linear equations in two variables, graphically and algebraically. (M10) **RF01** Students will be expected to model and solve problems that involve systems of linear inequalities in two variables. (M11)**	**RF07** Students will be expected to solve problems that involve linear and quadratic inequalities in two variables.	—

* M10—Mathematics 10
** M11—Mathematics 11

Background

In Mathematics 9, students solved single variable linear inequalities with rational coefficients and graphed their solutions on a number line (PR04). In Mathematics 11, students solved linear inequations and systems of linear inequations (RF01). Students are familiar with the inequality symbols, the terms **continuous** and **discrete data**, as well as their effect on a graph.

Students understand that the **boundary** for an inequality is the related equation. For example, the boundary for $x+2\geq 5$ is the equation $x + 2 = 5$ or $x = 3$. This boundary separates the number line into two sections. The solution set for the inequality is then determined as all values on one side of this boundary. To verify which side, students could select a point and determine if that point satisfies the equation.

In Mathematics 10, students graphed linear relations expressed in slope-intercept form, general form, and point-slope form (RF06) and in Mathematics 11, students solved systems of linear inequalities (RF02).

Students are familiar with verifying solutions to linear equations and linear inequalities using substitution. Students have been exposed to the concept that the solution of a linear inequality consists of a set of points while a linear equation has only one solution.

Solving a quadratic inequality in two variables implies finding, graphically, all the coordinate pairs (x, y) that make a particular inequality true.

The graph of a quadratic function separates the plane into two regions, one of which contains all the points that satisfy the inequality. Students will graph the corresponding function that is associated with the inequality and use test points to see which region should be shaded. They should continue to ask themselves if the graph should be drawn with a solid or broken curve.

Students are familiar with the convention of using a solid line to represent that the boundary is included in the solution set and a broken line to indicate that the boundary is not included in the solution set.

For this outcome it is expected that **test points** will be used to determine which region on the graph satisfies a linear or a quadratic inequality. A test point can be any point on either side of the boundary. If the inequality holds true, the test point is included as part of the solution.

For example, to graph the inequation $y > x^2 + 2x + 7$:

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$y = x^2 + 2x + 7$

or

$y = x^2 + 2x + 1 - 1 + 7$

$y = (x+1)^2 + 6$

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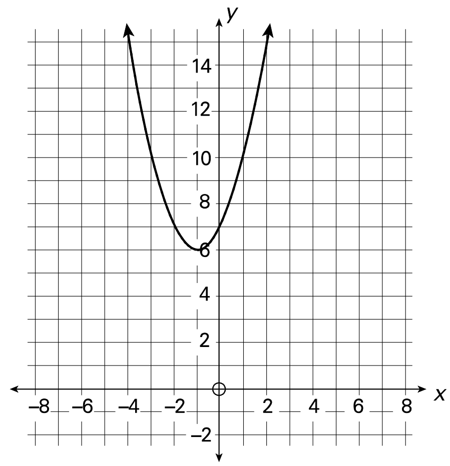

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<td>

Test the point (0, 8).

or

$8 > 0^2 + 2(0) + 7$

$8 > 7$

true

$\therefore$ Shade this region.

</td>
<td>

Test the point (0, 6).

$6 > 0^2 + 2(0) + 7$

$6 > 7$

false

$\therefore$ Do not shade this region.

Shade the other region.

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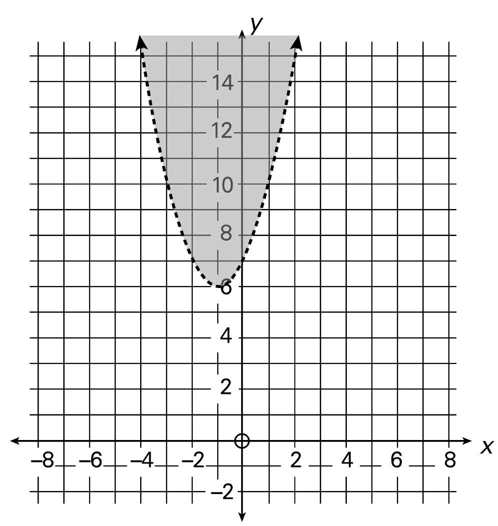
Sketching. The boundary will be broken (not included) and the region shaded will include the point (0, 8).



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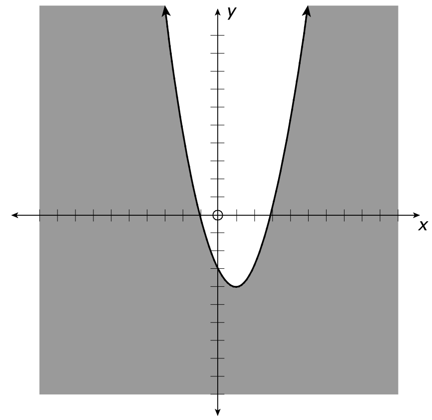


Students will also be able to write an inequality to describe a graph, given the function defining its boundary.

The **solution region** is the set of points that satisfy a linear or quadratic inequality, also known as the solution set.

For example: for $y \le x^2 - 2x - 3$, the quadratic function $y = x^2 - 2x - 3$ will mark the boundary and will be a solid line. Testing the point (0, –4), we determine that $-4 \le -3$ is a true statement and, therefore, the solution set will include all points in the same region as the point (0, –4) shown to the right as the shaded region and include the function.

In Mathematics 11, students worked with inequalities in real-life contexts (linear programming). They will continue to work with problems that can be expressed as an inequality in two variables requiring students to find two unknown quantities under certain constraints. Students will translate the word problem into an inequality. It is important to ensure that students understand and define the variables that are being used to represent the unknown quantities. Students should also understand that the shaded feasible region represents all possible combinations for the two quantities. An example, such as the following, could be used to highlight feasible and realistic solutions to a problem.

- With two minutes left in a basketball game, your team is 12 points behind. What are two different numbers of two-point and three-point shots your team could score to earn at least 12 points?

Students should understand that the domain and range contain only positive values. Ask them to identify realistic points in the feasible region considering the time remaining in the game.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Complete a table similar to the one below.

Inequality	Shade above or below	Broken or solid line
$y \le -2x + 8$		
$x + y > 2$		

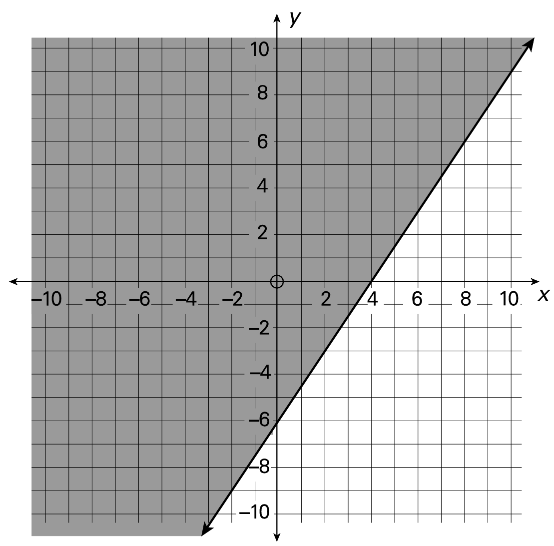
- A student was asked to graph $3x - 2y > 12$. His solution is shown below. Identify and correct his error(s).

  Student Solution:

  $3x - 2y > 12$

  $-2y > -3x + 12$

  $y > \frac{3}{2}x - 6$



- Your friend asks you to explain the difference between graphing $3y + 2x = 4$, $3y + 2x > 4$, and $3y + 2x \ge 4$. Write a response.

- The solution region for the inequality $5x - 3y > 10$ is above the line since it contains a "greater than sign." Do you agree or disagree with this statement? Explain the reasoning.

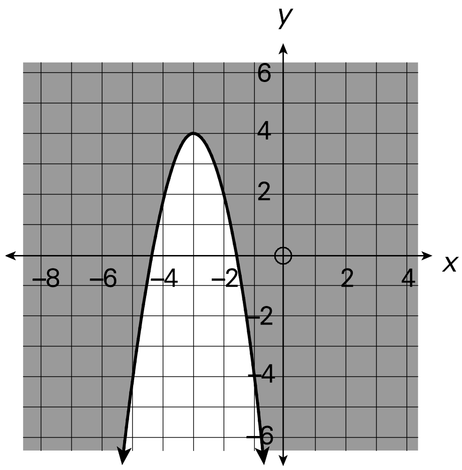
Whole-Class/Group/Individual Assessment Tasks

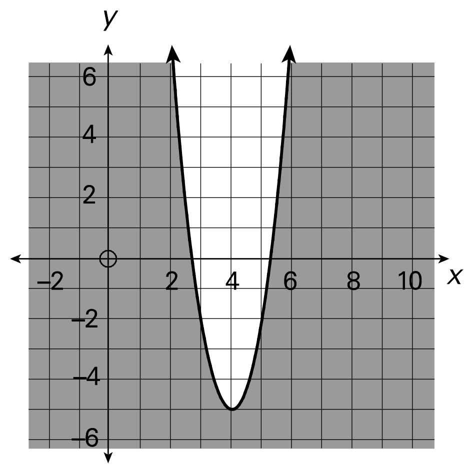
Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Graph $x^2 - 4x - 5 \ge y$ and determine, both from the graph and algebraically, if (−2, 4) is part of the solution. Is the boundary line solid or broken? Explain how you know.

- Sketch the solution region for each of the following inequalities.
  (a) $y \ge x^2 + 2x - 3$
  (b) $y > -2(x-1)^2 + 4$
  (c) $y < \frac{1}{3}(x-3)(x+3)$

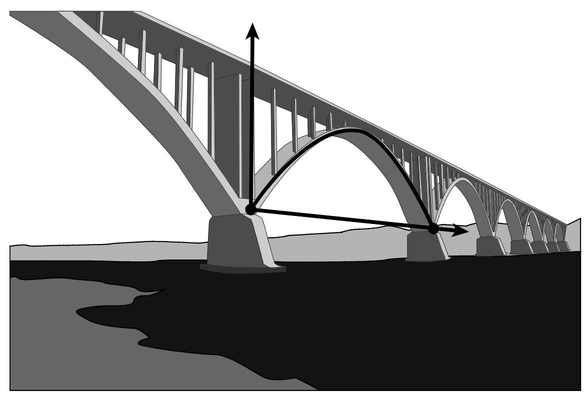
- Determine an inequality to match each graph.





- A fish jumps from the water and follows the trajectory described by the equation $d = -4.9t^2 + 4t$, where d is distance above the water in feet and t is time in seconds. Graph the function and determine the time interval when the fish is above the water.

- The Hugh John Flemming Bridge in Hartland, New Brunswick, is supported by several parabolic arches. The function $h(d) = -0.03d^2 + 0.84d - 0.08$ approximates the curve of one of the arches, where h represents the height above where the arch meets the vertical pier, and d represents the horizontal distance from the bottom left edge of the arch to the other end of the arch, both in metres.



- Write the inequality that describes the area under the curve. Is (4.5, 2.3) part of the solution set? Explain.

Follow-up on Assessment

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions

- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions

- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- For this outcome, students should begin with linear inequalities and then move on to quadratic inequalities, first graphing the line or curve of the corresponding equation as a solid ($\leq$, $\geq$) or broken line (<, >). The solution region should first be determined visually, and then confirmed by using a test point above or below the line or curve. If the inequality is true for the test point, then it is part of the solution region. This is an opportunity for students to make strong connections between algebraic and graphical solutions.

  Examples:
  - Susan plans to spend a maximum of 15 hours reviewing mathematics and biology in preparation for examinations. Draw a graph showing how much time she could spend studying each subject. (Review of Linear Inequalities completed in Mathematics 11.)
  - A contractor has at least one hundred tonnes of soil to be moved using two trucks. One truck has a four-ton capacity and the other has a five-ton capacity. Make a graph to show the various combinations of loads the two trucks could carry to complete the job. (Review of Linear Inequalities completed in Mathematics 11.)
  - Graph $3x - 2y < 12$ and determine, both from the graph and algebraically, if (−2, 4) is part of the solution. Is the boundary line solid or dashed? (Review of Linear Inequalities completed in Mathematics 11.)

- To be successful with graphing inequalities in two variables, students must be proficient with graphing linear equations, with and without technology. Identifying the form of the equation will help students decide which method they should choose when graphing the line (i.e., slope-intercept, point-slope, or *x*- and *y*-intercept method). Initially, students use test points to investigate the region of the plane that satisfies the inequality, which will help guide them to an understanding of when to shade above or below the boundary. Working through several examples should help them make the connection between the sign of the inequality, the shaded region, and whether to use a solid or broken line.

- Give students a number of linear and quadratic inequalities such as

  (a) $2x - 3y < 7$

  (b) $-x + 4y \geq -6$

  (c) $y > -2(x+1)^2 - 4$

  (d) $y \leq -x^2 + 1$

- Students can work in groups. Provide them with a list of five points and have them work together to determine which inequality, if any, satisfies the point.

- When solving single-variable inequalities, students sometimes forget to reverse the sign of the inequality when multiplying or dividing both sides by a negative number. A brief review and explanation of why this is the case may be necessary, as this error will also affect the solution when solving inequalities in two variables.

- Students should be provided with opportunities to write the equation of the linear or quadratic inequality given its graph. They will find the equation of the boundary line and then use the given shaded region to determine the correct inequality. Remind students what a broken or solid boundary line represents.

- A possible task that reinforces the graphs of inequalities is described below.

- Ask students to work in groups of two for this task. Each group should be given a deck of cards consisting of 12 with a graph, 12 with an inequality, and one “You win!” card. A dealer is chosen to deal the cards to each student. Students make pairs consisting of a graph and its matching inequality. These pairs are set aside face up on the table. Students take turns selecting a card from their partner and making matches if possible. Play continues until all matches are made. The student who has the YOU WIN card at the end is declared the winner.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- boundary
- inequality
- test points

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 9.1, pp. 464–475
  - Section 9.3, pp. 488–500

Notes

SCO RF08 Students will be expected to solve problems that involve quadratic inequalities in one variable. [CN, PS, V]			
[C] Communication **[T]** Technology	**[PS]** Problem Solving **[V]** Visualization	**[CN]** Connections **[R]** Reasoning	**[ME]** Mental Mathematics and Estimation

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF08.01 Determine the solution of a quadratic inequality in one variable, using strategies such as case analysis, graphing, roots and test points, or sign analysis; and explain the strategy used.

RF08.02 Represent and solve a problem that involves a quadratic inequality in one variable.

RF08.03 Interpret the solution to a problem that involves a quadratic inequality in one variable.

Scope and Sequence

Mathematics 10	Pre-calculus 11	Pre-calculus 12
RF09 Students will be expected to represent a linear function, using function notation.	**RF08** Students will be expected to solve problems that involve quadratic inequalities in one variable.	—

Background

In Mathematics 10, students were introduced to domain and range (RF01). They used interval notation and set notation to represent solution sets. This learning will now be extended to solving single variable quadratic inequalities.

This outcome follows from (RF07), with a numerical value being substituted for *y*. Given a value for *y*, the solution will be the values for *x* that satisfy the inequality. For example, for $x^2 - x - 2 > 0$, the solution will be the set of values of *x* for which $y > 0$.

It is important to have a discussion with students about what it means to solve a quadratic inequality in one variable. When solving $ax^2 + bx + c \geq 0$, for example, students should interpret this as finding the possible *x*-values where the corresponding *y*-values are zero or positive. In other words, when is the graph of $y = ax^2 + bx + c$ on or above the *x*-axis. This can be done graphically or algebraically. Students may have a better understanding of the algebraic method, however, if it is related back to the graph.

Solving quadratic inequalities in one variable involves finding the set of all *x*-values that made a particular inequality true.

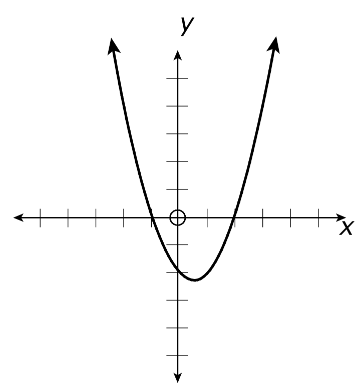
There are different methods used to solve quadratic inequalities in one variable, and students should be given the opportunity to practise the various methods to gain a deeper understanding of the concept and to find a method with which they are comfortable.

Most students will be comfortable with the graphing method. This method is very quick when the graph is known. In the case of quadratics this is likely to be the preferred method for most students. The sign analysis method is a very powerful method used in higher-level mathematics courses when the graph of the function of interest is not known. Similarly the roots method and case method are powerful methods used with many continuous functions when the graph may not be obvious. It is not essential that students learn to work with all of these methods.

Methods include the following:

The graphing method	requires students to sketch the graph of a function including the *x*-intercepts.
The roots method	requires students to plot the *x*-intercepts of a function and determine the *y*-coordinate of a point in each region created by these *x*-intercepts.
The sign analysis method	requires students to plot the *x*-intercepts of a function and determine the sign of the *y*-coordinate of a point in each region created by these *x*-intercepts.
The case method	requires students to consider all the possible situations (e.g., if a quadratic is greater than zero, this means that either both its factors are positive or both its factors are negative).

- **Method 1: Graphing Method**
  Find all solutions for $(x-2)(x+1)>0$.
  The roots of the corresponding equation are $x = 2$ or $x = -1$, and represent where the function $f(x)=(x-2)(x+1)$ is equal to zero. These would be the *x*-values where the function would change sign from positive to negative or from negative to positive.

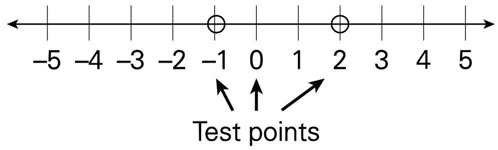
  

  Visually, students know that the equation is that of a quadratic that opens up, and therefore, they can see that the function $y=x^2-x-2$ is greater than zero when $x<-1$ or $x>2$.

  For $x^2-x-2>0$, the solution is
  $\{x \mid x<-1 \text{ or } x>2, x\in R\}$ or $(-\infty,-1)\cup(2,+\infty)$.

- **Method 2: Roots and Test Points Method**
  Find all solutions for $(x-2)(x+1)>0$.
  The roots of the corresponding equation $(x-2)(x+1)=0$ are $x = 2$ and $x = -1$, define the boundary for the inequality and three different sections of the number line.

  These *x*-intercepts can be shown on a single number line. The *x*-intercepts divide the number line into regions. A test point from each region can be substituted into the inequality. If the value satisfies the inequality, that region is part of the solution set. Since a quadratic is a continuous curve, the only places where the quadratic will change from positive to negative or from negative to positive will be at the roots of the quadratic. It follows that all points in a region will have the same sign, and therefore, testing a single point is sufficient to establish the entire region as the solution.



$(-2-2)(-2+1)>0$	$(0-2)(0+1)>0$	$(5-2)(5+1)>0$
$(-4)(-1)>0$	$(-2)(1)\not>0$	$(3)(6)>0$
$4>0$	$-2\not>0$	$18>0$

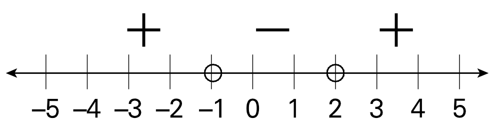
The points in the outside regions (–2 and 5) satisfy the inequality (give an answer > 0), but the point in the centre region (0) does not. Therefore the solution set lies in the outside regions:
$\{x \mid x<-1 \text{ or } x>2, x\in R\}$ or $(-\infty,-1)\cup(2,+\infty)$.

(Note that –1 and 2 are open circles to indicate that these roots are not included in the solution. If the sign as $\geq$ instead of $>$, the circles would be filled in to indicate that they were included in the solution.)

- **Method 3: Sign Analysis Method**
  Find all solutions for $(x-2)(x+1)>0$.

  For solving quadratic inequalities, the key piece of information is identifying the region(s) in which the function is positive and the region(s) in which the function is negative. In this example, the solution set must be the *x*-values for which the function is positive.

  As with the previous method, roots are determined and test points from each region are substituted into the inequality to determine the sign of the whole region. For this method, specific values of the *y*-coordinate are not necessary. It is only necessary to determine if the *y*-value is positive or if it is negative. If the quadratic had no *x*-intercepts then the quadratic would either be always positive or always negative. If the quadratic's vertex was its *x*-intercept, then the quadratic would be always greater than or equal to zero or always less than or equal to zero.



The outside regions satisfy the inequality so the solution is
$\{x \mid x<-1 \text{ or } x>2, x\in R\}$ or $(-\infty,-1)\cup(2,+\infty)$.

- **Method 4: Case Method**
  Find all solutions for $(x-2)(x+1)>0$.

  When two factors multiply to give a positive result both factors must be either positive or both must be negative. These two cases must be considered.

  $(x-2)>0$ and $(x+1)>0$
  $x>2$ and $x>-1$

  Stating that the value of x must be both greater than 2 and also greater than –1 is the same as saying that it must be greater than 2.

  $(x-2)<0$ and $(x+1)<0$
  $x<2$ and $x<-1$

  Stating that the value of x must be both less than 2 and also less than –1 is the same as saying that it must be less than –1.

The outside regions satisfy the inequality so the solution is
$\{x\mathrm{I}x<-1 \text{ or } x>2, x\in R\} x<-1$ or $x>2, \in R$ or $(-\infty,-1)\cup(2,+\infty)$.

Regardless of the technique used, proficiency in solving quadratic equations is important.

Students will solve problems that involve a quadratic inequality in one variable. These types of problems will be similar to solving a quadratic equation except there is usually a minimum or maximum constraint. Students should relate their solution to the context of the problem.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

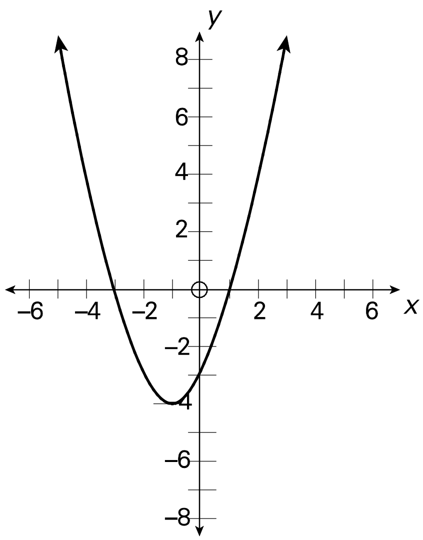
- Is the value of the expression $3x^2-5x$ positive or negative for $x=-2$?

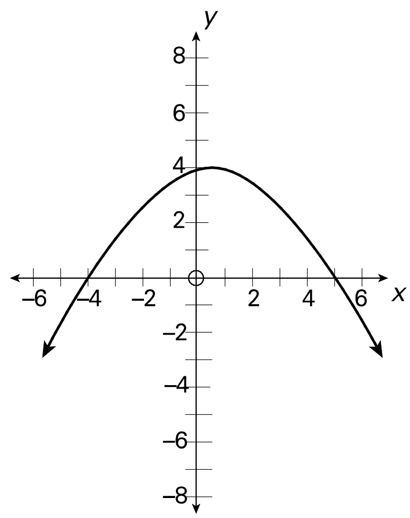
- Solve the equation.
  (a) $3x^2-14x-5=0$
  (b) $(4x-1)(x+3)=0$
  (c) $2(x-1)^2-6=0$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Given $y_1 = x^2 - 7$ and $y_2 = 5 - 4x$, for what values of x is $y_1 < y_2$?
- Use the graphs shown below to state the x-values for which $f(x) > 0$.



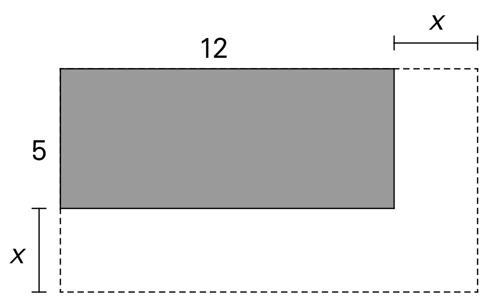


- When a projectile is fired into the air, its height h (in metres) t seconds later is given by the equation $h(t) = 11t - 3t^2$.
  (a) When is the projectile at least 6 m above the ground?
  (b) How is your answer for part (a) related to the graph of $h(t) = 11t - 3t^2$?
  (c) How is your answer for part (a) related to the graph of $h(t) = 11t - 3t^2 - 6$?
- When a baseball is hit by a batter, the height of the ball, $h(t)$, at time t, is determined by the equation $h(t) = -16t^2 + 64t + 4$. For which interval of time is the height of the ball greater than or equal to 52 feet?
- The surface area, A, of a cylinder with radius r is given by the formula $A = 2r^2 - 5r$. What possible radii would result in an area that is greater than 12 cm^2?
- Your friend asks you to explain the difference between solving $x^2 - 2x - 3 = 0$, $x^2 - 2x - 3 \geq 0$, and $x^2 - 2x - 3 < 0$. Write a response.
- Give an example of a quadratic inequality that would have a single-value solution.
- Give an example of a quadratic inequality that would have no solution.

- Solve the following algebraically and illustrate graphically.
  (a) $2x^2 - 5x - 7 \le 0$
  (b) $(3x-1)(x+4) > 0$
  (c) $3(x-1)^2 - 6 \ge 0$
  (d) $x^2 - 8x + 10 < 0$

- A chef determines that she can use the formula $P \le 5 - m^2$ to estimate when the price of flour will be P dollars per kilogram or less in m months from the present.
  (a) When will flour be available for $2/kg or less?
  (b) Explain why some of the values of m that satisfy the inequality do not make sense in this context.
  (c) Write and solve the inequality if the flour is $1/kg or less?

- The base of a rectangular bin currently has dimensions 12 *m* × 5 *m*. The base is to be enlarged by an equal amount on the width and length so that the area is more than doubled. Ask students, By how much should the length and width be increased to produce the desired area?



FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students should learn how to check their solution to one-variable quadratic inequalities using their graphing calculators. This provides a quick check for their algebraic methods.

- This section can be introduced with a question such as,
  The observed rabbit population on an island is given by the function $P(t) = -0.4(t-42)^2 + 1200$, where t is the time in months since they began observing the rabbits. For how many months is the rabbit population in excess of 950?
  (a) When is the rabbit population more than 950?
  (b) How is your answer for part (a) related to the graph of $P(t) = -0.4(t-42)^2 + 1200$?
  (c) How is your answer for part (a) related to the graph of $P(t) = -0.4(t-42)^2 + 250$?

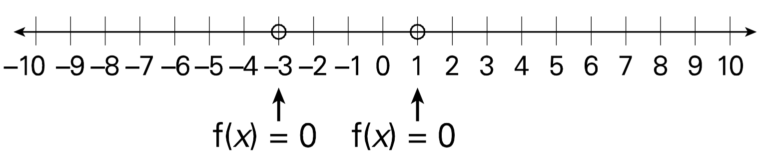
- One strategy to solve an inequality that students should be familiar with involves plotting roots on a number line and testing points in each interval. Many students will not want to bother to test values in each of the intervals since once they determine the x-intercepts, they will know where the quadratic is positive and where it is negative by applying their knowledge of its direction of opening.

  This non-graphing method is very important and will be used to solve inequalities in higher-level mathematics courses, so it is important that students understand its underlying premise.

  For example, solving an inequality such as $x^2 + 2x - 3 < 0$, students first determine the roots of the quadratic equation $x^2 + 2x - 3 = 0$.
  $x^2 + 2x - 3 = 0$
  $(x+3)(x-1) = 0$
  $x+3 = 0;\ x-1 = 0$
  $x = -3;\ x = 1$

- These values represent the x-values where the quadratic equals zero. For any function that does not have any gaps or breaks (is **continuous**) the x-intercepts will represent the only x-values where it is possible for the function to change sign. Since quadratics are **continuous functions**, we can conclude that in the intervals less than –3, between –3 and 1, and greater than 1, the sign of the quadratic will not change. For this reason, only one value in each of these intervals needs to be tested to determine if it is positive or negative.

  A sign diagram consisting of a number line, the x-intercepts, and testing point in each interval are used to determine the intervals that satisfy the inequality.

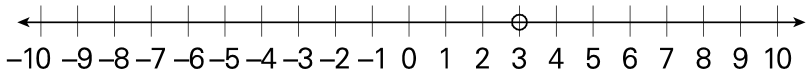


  Test some x-value less than –3 to determine if it satisfies the inequation $x^2 + 2x - 3 < 0$.
  Test some x-value between –3 and +1 to determine if it satisfies the inequation $x^2 + 2x - 3 < 0$.
  Test some x-value greater than +1 to determine if it satisfies the inequation $x^2 + 2x - 3 < 0$.

Relating the sign diagram to the *x*-axis of the graph, students use test points to determine if the function is positive or negative. They should understand that the roots –3 and 1 are not part of the solution. Students can write the solution set as set notation or interval notation.

If the quadratic has one *x*-intercept, $x^2 - 6x + 9 > 0$.

$x^2 - 6x + 9 = 0$
$(x - 3)(x - 3) = 0$
$x - 3 = 0$
$x = 3$



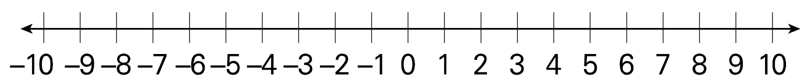
Test a value less than 3.
$x^2 - 6x + 9 = 0$
$(2)^2 - 6(2) + 9 > 0$
$4 - 12 + 9 > 0$
True.

Test a value greater than 3.
$x^2 - 6x + 9 = 0$
$(4)^2 - 6(4) + 9 > 0$
$16 - 24 + 9 > 0$
True.

$\therefore \{x : x < 3 \text{ or } x > 3,\ x \in R\}$ or $(-\infty, 3) \cup (3, \infty)$

If the quadratic has no *x*-intercepts, $x^2 + 9 \geq 0$.

$x^2 + 9 = 0$
$x^2 = -9$
Not possible.



This quadratic has no *x*-intercepts; therefore, it is always above the *x*-axis or always below the *x*-axis.

Test any value.
$x^2 + 9 \geq 0$
$(2)^2 + 9 \geq 0$
$4 + 9 \geq 0$
True.
$\{x : x \in R\}$ or $(-\infty, \infty)$.

If the question had been $x^2 + 9 \leq 0$, then there would have been no solution.

SUGGESTED MODELS AND MANIPULATIVES

- grid paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- continuous function
- sign analysis
- interval notation
- set notation

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 9.2, pp. 476–487

Notes

SCO RF09 Students will be expected to analyze arithmetic sequences and series to solve problems. [CN, PS, R, T]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF09.01 Identify the assumption(s) made when defining an arithmetic sequence or series.
RF09.02 Provide and justify an example of an arithmetic sequence.
RF09.03 Derive a rule for determining the general term of an arithmetic sequence.
RF09.04 Describe the relationship between arithmetic sequences and linear functions.
RF09.05 Determine t_1, d, n, or tn in a problem that involves an arithmetic sequence.
RF09.06 Derive a rule for determining the sum of n terms of an arithmetic series.
RF09.07 Determine t_1, d, n, or Sn in a problem that involves an arithmetic series.
RF09.08 Solve a problem that involves an arithmetic sequence or series.

Scope and Sequence

Mathematics 10 / Mathematics 11	Pre-calculus 11	Pre-calculus 12
RF07 Students will be expected to determine the equation of a linear relation to solve problems, given a graph, a point and the slope, two points, and a point and the equation of a parallel or perpendicular line. (M10)* **RF09** Students will be expected to represent a linear function, using function notation. (M10) **LR01** Students will be expected to analyze and prove conjectures, using inductive and deductive reasoning, to solve problems. (M11)**	**RF09** Students will be expected to analyze arithmetic sequences and series to solve problems.	—

* M10—Mathematics 10
** M11—Mathematics 11

Background

In Mathematics 9, students were exposed to linear patterns (PR01). In Mathematics 10, they determined the equation of a linear relation (RF07).

Students are expected to use the words *sequence* and *series* correctly. A **sequence** is an ordered list of elements; a **series** is the sum of a sequence of numbers.

For this outcome, students will explore the concept of arithmetic sequences and series. They will work with patterns where there is a **common difference** between consecutive terms. The common difference, d, is the difference between successive terms in an arithmetic sequence. Students will write a formula for the **general term**, an expression determining any term of a sequence, of an arithmetic sequence and solve for any missing values.

When solving problems involving arithmetic sequences, students will consider a variety of examples such as, but not limited to, the following:

- Find the general term t_n using tables, charts, graphs, or an equation.
- Find the number of terms in a finite arithmetic sequence when given the value of the nth term.
- Find the common difference when provided with algebraic expressions representing the value of terms for the sequence.

Once sequences have been explored, students will be introduced to an arithmetic series. They will make the connection that an arithmetic series is the sum of an arithmetic sequence, derive a formula, and apply it in a variety of problem-solving situations.

Students will evaluate and/or manipulate the given arithmetic series to determine the first term, the common difference, the number of terms in the sequence, the nth term, or the sum. They should solve problems that involve an arithmetic series within a context.

After investigation of several sequences that are arithmetic, students should conclude that if a sequence is arithmetic, the same number is being added to each term in the sequence.

Students are expected to investigate the concept of a sequence by observing a variety of number patterns to develop an understanding of the notation, symbols, and domain associated with arithmetic sequences.

Specifically, students should understand that

- d represents the common difference of an arithmetic sequence
- t_1 [some resources use u_1] represents the first term of any sequence
- t_2 [some resources use u_2] represents the second term of any sequence
- t_n [some resources use u_n] represents the nth term of any sequence
- S_1 represents the sum of the first term of any sequence
- S_2 represents the sum of the first two terms of any sequence (this value would be the second term of the series)
- S_n represents the sum of the first n terms of any sequence (this value would be the nth term of the series)

In Mathematics 10, students were exposed to the concept of domain as it relates to linear functions (RF01).

Students will make the connection that arithmetic sequences, when graphed, form linear graphs. Students may need to review graphing linear functions using a table of values and/or the slope y-intercept method. They should observe the following features when comparing an arithmetic sequence to a linear function.

- The slope is the common difference.
- The general term of an arithmetic sequence is linear.
- The y-intercept is the initial value minus the common difference $(t_1 - d)$.

- The domain of a sequence is the set of natural numbers, while the domain of a linear function, based on the context, can be the set of real numbers.
- The graph of an arithmetic sequence consists of discrete data, while the graph of a linear function may be discrete or continuous.

The following formulas should be developed with the students as well as understood and applied efficiently by students.

$$d = t_n - t_{n-1}$$

$$t_n = t_1 + (n-1)d$$

$$S_n = \frac{n}{2}(t_1 + t_n)$$

$$S_n = \frac{n}{2}[2t_1 + (n-1)d]$$

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Explain the difference between discrete and continuous data.
- What are the natural numbers?
- Evaluate the following:
  - $u(t) = 3t + 4$, determine $u(3)$.
  - $t(x) = 5x - 11$, find the value of x that makes $t(x) = 9$.

- While surfing the Internet, you find a site that claims to offer "the most popular and the cheapest DVD anywhere." Unfortunately, the website is not clear about how much they charge for each DVD, but it does give you the following information:

Number of DVDs ordered	1	2	3
Total cost (includes shipping and handling)	\$15	\$24	\$33

  Plot the data shown in the table, and write an equation for the relation.

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Determine the first five terms for each of the sequences.
  (a) $t_n = 2n - 1$
  (b) $t_n = \sqrt{4n^2 - 4n + 1}$
  (c) $t_n = \dfrac{2n^2 + n - 1}{n + 1}$
  (d) Which of the sequences, generated by the formulas, are arithmetic? Explain your reasoning.

- Determine whether each of the following sequences is arithmetic. Justify your conclusions.
  (a) 5.3, 5.9, 6.5
  (b) $x - 1$, $x + 1$, $x + 3$
  (c) x^1, x^2, x^3
  (d) $2x + 5$, $4x + 5$, $6x + 5$

- Create two arithmetic sequences and find the general term t_n. Determine the sum and then the product of the two sequences. Explain if these new sequences created are arithmetic and justify your reasoning.

- Determine the next four terms in the arithmetic sequence –5, –2, 1, … Write a rule for the general term, t_n , of this arithmetic sequence.

- Find the first term in the arithmetic sequence for which $t_{19} = 42$ and $d = -\dfrac{2}{3}$.

- Find d for the arithmetic sequence in which $t_1 = 6$ and $t_{14} = 58$.

- Find the 47th term in the arithmetic sequence –4, –1, 2, 5, …

- How many terms are there in the arithmetic sequence 40, 38, 36, …, –30?

- In an arithmetic sequence, $t_{12} = 52$ and $t_{22} = 102$, find the first three terms of the sequence and a general formula for t_n.

- Write an arithmetic sequence that has 4.9 as the first term, 2.5 as the last term, and five other terms in between these two terms.

- Antonio has found that he can input statistical data into his computer at a rate of 2 data items faster each half hour he works. One Monday, he starts work at 9:00 a.m., inputting at a rate of 3 data items per minute. At what rate will Antonio be inputting data into the computer at 11:30 a.m.?

- How many multiples of 6 are there between 65 and 391?

- For each arithmetic sequence, ask students to write a formula for t_n and use it to find the indicated term.
  (a) –4, 1, 6, 11, … , t_{13}.
  (b) 9, 1, –7, –15, …, t_{46}.

- Consecutive terms of an arithmetic sequence are $(5 + x)$, 8, and $(1 + 2x)$. Determine the value of x.

- The diagram shows a pattern of positive integers in five columns. If the pattern is continued, determine the columns in which the numbers indicated would appear.

  (a) 49
  (b) 117
  (c) 301
  (d) 8725

Col 1	Col 2	Col 3	Col 4	Col 5
1		2		3
	5		4	
6		7		8
	10		9	
11		12		13
	15		14	

- Consider the following problem and explain if the calculation is correct. Justify your reasoning.
  - Paul must determine the 50th term of an arithmetic sequence beginning with 5 and having a common difference of 9. He calculates (50 × 9) + 5 = 455.

- Zachary is having trouble remembering the formula $t_n = t_1 + (n-1)d$ correctly. He thinks the formula should be $t_n = t_1 + nd$. How would you explain to Zachary that he should use $(n – 1)d$ rather than nd in the formula.

- Use the pattern to do the following tasks: 4, 7, 10, 13, 16, ...
  (a) Find a formula for the general term t_n and record your findings.
  (b) Graph the discrete data.
  (c) Draw a line through the points and record the equation.
  (d) Compare your results from steps (a) and (c). How does the common difference relate to the slope?
  (e) Subtract d from the first term of the sequence. Compare this value to the y-intercept from the line of best fit. What do you notice?

- Explain how you could quickly determine the sum of
  (a) the first 12 positive integers
  (b) the first 10 even integers
  (c) the first 100 odd integers

- A round robin tournament is to be played by 3 teams. Determine the total number of games required to guarantee that each team plays the other exactly once. Repeat the same task for a tournament consisting of 4 teams. Use your results from above to find the total number of games for a national round robin tournament with all ten provinces and three territories participating. Explain how your answer is related to an arithmetic series.

- Find the sum of 20 + 14 + 8 + ... + (–70).

- A theatre has 60 seats in the first row, 68 seats in the second row, 76 seats in the third row, and so on in the same increasing pattern. If the theatre has 20 rows of seats, how many seats are in the theatre?

- Find S_{30} for the arithmetic series –2 + 4 +10 + ...

- Find the sum of each of the following arithmetic series.
  (a) 9 + 14 + 19 + ... + 304
  (b) 24 + 18 + 12 + ... –36

- A theatre has 24 rows of seats. The front row has 5 seats on each side of the centre aisle. Each successive row has one more seat on each side. How many seats are there altogether?

- The Arroyos are planning to build a brick patio that approximates the shape of a trapezoid. The shorter base of the trapezoid needs to start with a row of 5 bricks, and each row must increase by two bricks on each side until there are 25 rows. How many bricks do the Arroyos need to buy?

- An apprentice is hired at a starting salary of $1000 per month. Each subsequent month for the first year, her salary increases by $150. How much does she earn for the whole year?

- Bags of pellets are stored in piles. The bottom layer of the pile has 20 bags along its length and is 4 bags wide. Each layer above has one less bag lengthwise than the previous layer but has the same width. The top layer has 16 bags in total. How many bags of pellets are in the pile?

- The sum of the first 8 terms of an arithmetic series is 52 and the sum of the first 16 terms of the same series is –88. Determine the value of the first term and the common difference.

- In an arithmetic series t_1 = 6 and S_9 = 108. Find the common difference and the sum of the first 20 terms.

- For three months in the summer (12 weeks), Job A pays $325 per month with a monthly raise of $100. Job B pays $50 per week with a weekly raise of $10. Which is the better-paying job and why.

- Show that $1 + 3 + 5 + ... + (2n – 1) = n^2$.

- Show that the sum of any arithmetic series is a quadratic.

- Is it possible for an arithmetic series to have $S_{12} = S_{26}$?

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Encourage students to provide their own examples of arithmetic sequences and justify, finding the difference between consecutive terms, why such a sequence is arithmetic. Students should conclude, that if a sequence is arithmetic the same number is being added to each term in the sequence.

- Use a numerical example to give students an opportunity to identify patterns and a possible rule for an arithmetic sequence. Intuitively students will see that a common difference is being added to obtain each successive term of an arithmetic pattern. This is called a recursive description, and it works really well for using any term to get the successive term, it is less successful for using the first few terms to get the 100th term.

  For example, the sequence 5, 7, 9, 11, 13, ... could be described as "add 4 to each term to get the next term."

  In order to obtain the functional, rather than recursive, way of describing an arithmetic sequence, ask students to consider a sequence such as 5, 7, 9, 11, 13, ... and use the following questions to guide student discussion.
  - What is the common difference?
  - Can this sequence be rewritten to show the pattern of the first term and the common difference?

- Can you predict the formula for the general term of an arithmetic sequence based on the pattern 5, 5 + 2, 5 + 2(2), 5 + 3(2), ... ?
- Can you write the pattern in general terms for any first term and common difference?

- Ensure that students are able to easily work with sequence and series notation. They must understand what the symbols t_1 , t_n , S_n and n represent.

- For some students, using subscripts is difficult; they may find it easier to use functional notation. For example, they could use $t(n)$ rather than t_n. Rather than have the students' work look confusing, encourage them to use the notation that they find easiest to write and to read.

- It is likely a good idea to vary the notation used so that students see the general term written in a variety of forms such as t_n , u_n , and a_n.

- Remind students that the value of n must be a natural number; that is, n can only take on the values 1, 2, 3, 4, ...

- Students should notice that the terms of the arithmetic sequence with first term t_1 and a common difference d can be written as

  $t_1 = t_1$
  $t_2 = t_1 + d$
  $t_3 = t_1 + 2d$
  $t_4 = t_1 + 3d$
  …

  Using this pattern, students can write the formula for any arithmetic sequence $t_n = t_1 + (n-1)d$.

- Remind students that if the value of d is positive, then the arithmetic sequence is increasing, and if the value of d is negative, then the arithmetic sequence is decreasing.

- Give students an opportunity to write their own examples of arithmetic sequences and to write the appropriate formula. They should use numerical and algebraic examples. Encourage them to list t_1, d, and n, and then write the equation for t_n.

- When provided with the general term, students sometimes have difficulty differentiating between the term and the term number. When they are asked to find the tenth term, for example, they may be unsure whether to write t_{10} or $t_n = 10$. It is important to reinforce that t_{10} represents the tenth term while $t_n = 10$ represents the nth term having a value of 10.

- The relationship between an arithmetic sequence and a linear function can be highlighted through a discussion such as the following:
  - Find the general term t_n. Compare this equation to a linear equation $y = mx + b$. Are they similar?
  - Graph the sequence. What do you notice?
  - What is the relationship between the slope of the line and the common difference?
  - Evaluate the first term minus the common difference of the sequence. What does this value represent in the linear equation?

- The relationship between arithmetic sequences and linear functions could also be developed using a symbolic approach.

$$t_n = t_1 + (n-1)d$$
$$t_n = t_1 + dn - d$$
$$t_n = dn + t_1 - d$$
$t_n = dn + (t_1 - d)$ compared to $y = mx + b$
where $d = m$ and $t_1 - d = b$

- The following example provides an opportunity to review arithmetic sequences and introduce arithmetic series.
  - Suppose you create a Facebook account and add one new friend on day 1, four more friends on day 2 and four more friends each day for a total of 25 days. How many friends would you have in total after the 25th day?

  - Use the following questions to guide students:
    > What is the arithmetic sequence?
    > What is the first term?
    > What is the rule for t_n? What is the last term?
    > What does the sum of the terms in the sequence represent?
    > How many terms are we summing up?
    > How can the value of S_{25} be determined?

    (Note: Although some students may just add up all the terms in the sequence, they should understand this is not the most efficient method.)

- Introduce students to Gauss's method of determining the sum of the first 100 positive integers. Use this method to determine S25 in the previous example. Ask students to write the series twice, once in ascending order and the other in descending order. Then, find the sum of the two series.

$$\begin{array}{l} S_{25} = 1+5+9+\ldots+89+93+97 \\ \underline{+S_{25} = 97+93+89+\ldots+9+5+1} \\ 2S_{25} = 98+98+98+\ldots+98+98+98 \\ 2S_{25} = 25(98) \\ S_{25} = \dfrac{25}{2}(98) \end{array}$$

Encourage students to try to determine where the different parts of that formula come from. When they are trying to make a conjecture about the general case, it is best to leave the original values rather than simplifying the expression. Ask students what they think the values 25 and 98 represent. They should recognize that 25 is the number of terms and 98 is the sum of 1 and 97, representing t_1 and t_{25}. Therefore, $S_{25} = \dfrac{25}{2}(1+97)$.

The algebraic approach should also be explored to derive the formula for the sum of the general arithmetic series. The sum of n terms of an arithmetic series is represented by

$$S_n = t_1 + t_2 + \ldots + t_{n-1} + t_n$$
$$S_n = t_1 + (t_1 + d) + \ldots + (t_1 + (n-2)d) + (t_1 + (n-1)d)$$

Using Gauss's method, guide students through the following derivation:

$$S_n = t_1 + t_1 + d + \ldots\ t_1 + (n-2)d + t_1 + (n-1)d$$
$$S_n = t_1 + (n-1)d + t_1 + (n-2)d + \ldots\ t_1 + d + t_1$$
$$2S_n = 2t_1 + (n-1)d + 2t_1 + (n-1)d + \ldots\ 2t_1 + (n-1)d + 2t_1 + (n-1)d$$
$$2S_n = n(2t_1 + (n-1)d)$$
$$2S_n = n(t_1 + \underbrace{t_1 + (n-1)d}_{t_n})$$
$$2S_n = n(t_1 + t_n)$$
$$S_n = \frac{n(t_1 + t_n)}{2}$$

- Show how to derive each version of the formula for the sum of an arithmetic series from the other version.

- When dealing with questions in context, some students will experience difficulty determining if the pattern is arithmetic or not, and whether a series or a sequence is being called for.

- Consider the following examples:
  - An auditorium has 20 seats on the first row, 24 seats on the second row, 28 seats on the third row, and so on, and has 30 rows of seats. How many seats are in the auditorium?

Some steps that will help students are as follows:

> Strongly suggest to students that they write out the pattern that is evident in the question and then clearly define the meaning of the first few terms of the pattern that they have written.

20, 24, 28, 32, ...
20 represents the number of seats in the first row.
24 represents the number of seats in the second row, etc.
This pattern is arithmetic since it has a common difference.

> Determine what exactly the problem is asking, and decide how it fits into the pattern you wrote.

20, 24, 28, 32, ..., ???
The 30th term would represent the number of seats in the 30th row.
We want *all* the seats in the auditorium, and so will need to add up the seats in all 30 rows; therefore, we will need to find S_{30}.

> Students then need to think about what information they actually know and use it to obtain the answer to the question.

$t_1 = 20$
$d = 24 - 20 = 4$
$n = 30$
Find S_{30}.

$$S_n = \frac{n}{2}[2t_1 + (n-1)d]$$
$$S_{30} = \frac{30}{2}[2(20) + (30-1)4]$$
$$S_{30} = 15[40 + 116] = 2340$$

– A stadium has 5250 people in it. If the people are leaving the stadium at a rate of 210 people per minute, how long does it take for there to be less than 400 people remaining in the stadium?

Some steps that will help students are as follows:

> Strongly suggest to students that they write out the pattern that is evident in the question and then clearly define the meaning of the first few terms of the pattern that they have written.

5250, 5040, 4830, 4620, ...
5250 represents the number of people in the stadium before they start to leave.
5040 represents the number of people in the stadium after one minute has passed.
4830 represents the number of people in the stadium after two minutes has passed.
This pattern is arithmetic since it has a common difference.

Determine what exactly the problem is asking and decide how it fits into the pattern you wrote. 5250, 5040, 4830, 4620, ..., 400
We need to find what term is 400. Therefore, we need to solve for n when t_n = 400.

You may wish to recommend that your students let t_1 = 5040 since that term is the number of people in the stadium after one minute; therefore, the term number easily relates to the meaning of the term. This makes it easier to understand the meaning of the answer when the question is solved.

> Students then need to think about what information they actually know and use it to obtain the answer to the question.

$t_1 = 5040$
$t_n = 400$
$d = -210$
Find n.

$$t_n = t_1 + (n-1)d$$
$$400 = 5040 + (n-1)(-210)$$
$$-4640 = (n-1)(-210)$$
$$22.1 = n - 1$$
$$23.1 = n$$

Since n must be a natural number, the answer n = 23.1 is not possible as a term number. It does, however, tell us that the number of people in the stadium reaches 400 after a bit more than 23 seconds has passed. By the time 24 seconds have passed, the number or people remaining is less than 400.

- It is possible to have a arithmetic series with a negative constant difference such that it has a specific sum at two different times. For example 100 + 95 + 90 + 85+ ... t_n = 595. Determining the value of *n* requires the solution of a quadratic.

$$100+95+90+85+...+t_n=595$$
$$t_1=100$$
$$d=-5$$
$$S_n=595$$
$$S_n=\frac{n}{2}(2t_1+(n-1)d)$$
$$595=\frac{n}{2}(2(100)+(n-1)(-5))$$
$$595=\frac{n}{2}(200-5n+5)$$
$$1190=n(205-5n)$$
$$1190=-5n^2+205n$$
$$5n^2-205n+1190=0$$

$$5n^2-205n+1190=0$$
$$n=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$
$$n=\frac{205\pm\sqrt{(-205)^2-4(5)(1190)}}{2(5)}$$
$$n=\frac{205\pm\sqrt{18225}}{10}$$
$$n=\frac{205\pm135}{10}$$
$$n=340 \text{ or } n=7$$

Therefore 7 terms add to 595 and 340 terms add to 595.

SUGGESTED MODELS AND MANIPULATIVES

- counting disks
- Cube-A-Links
- graph paper

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- arithmetic sequence
- arithmetic series
- common difference
- explicit notation
- general term
- sequence
- series

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 1.1 and 1.2, pp. 6–31

Notes

SCO RF10 Students will be expected to analyze geometric sequences and series to solve problems. [PS, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF10.01 Identify assumptions made when identifying a geometric sequence or series.
RF10.02 Provide and justify an example of a geometric sequence.
RF10.03 Derive a rule for determining the general term of a geometric sequence.
RF10.04 Determine t_1, r, n, or tn in a problem that involves a geometric sequence.
RF10.05 Derive a rule for determining the sum of n terms of a geometric series.
RF10.06 Determine t_1, r, n, or Sn in a problem that involves a geometric series.
RF10.07 Generalize, using inductive reasoning, a rule for determining the sum of an infinite geometric series.
RF10.08 Explain why a geometric series is convergent or divergent.
RF10.09 Solve a problem that involves a geometric sequence or series.

Scope and Sequence

Mathematics 10	Pre-calculus 11	Pre-calculus 12
AN03 Students will be expected to demonstrate an understanding of powers with integral and rational exponents.	**RF10** Students will be expected to analyze geometric sequences and series to solve problems.	**RF09** Students will be expected to graph and analyze exponential and logarithmic functions. **RF10** Students will be expected to solve problems that involve exponential and logarithmic equations.

Background

For this outcome, students will continue to explore patterns with a focus shifting to geometric sequences and series. Upon completion of this outcome, it is expected that students will be able to identify a sequence and a series as being arithmetic, geometric, or neither.

Students will explore examples of geometric sequences having a “common ratio” pattern resulting in a specific formula. Similar to an arithmetic sequence, students will continue to use the same notation when working with the terms of a geometric sequence. The symbol r will represent the common ratio. With teacher support, students will determine that the general term of a geometric sequence is $t_n = t_1(r)^{n-1}$.

A geometric sequence can be written recursively or as a function. The focus of this course is on representing it as a function. For example, the sequence 3, 6, 12, 24, ... can be described recursively as "doubling each term to obtain a successive term."

In Mathematics 10, students worked with linear equations and were able to recognize in 10RF09 that arithmetic sequences were linear in nature. While students have worked with exponents, they have not had any experience with the graphs or the equations of exponential functions.

- **geometric sequence**—A sequence in which the ratio of consecutive terms is constant.
- **common ratio**—The ratio of successive terms in a geometric sequence, $r = \frac{t_n}{t_{n-1}}$ may be positive or negative; for example, in the sequence 3, 6, 12, 24, ..., the common ratio is 2.

Once students have had experience with geometric sequences, they will then consider geometric series. Students should develop, with summation of the terms of the general geometric sequence, the two formulas: $S_n = \frac{t_1(r^n - 1)}{r-1}; r \neq 1$ and $S_n = \frac{t_1(1-r^n)}{1-r}; r \neq 1$.

- **geometric series**—The terms of a geometric sequence expressed as a sum; for example, 3 + 6 + 12 + 24 is a geometric series.

  Initially, students will work with finite geometric sequences and series, evaluating and/or manipulating the given geometric series to determine
  - the first term
  - the common ratio
  - the number of terms in the sequence
  - the *n*th term or the finite sum

- Solve contextual problems involving a geometric series.

Students will be expected to differentiate between a convergent and divergent series.

- **convergent series**—A series with an infinite number of terms in which the sequence of partial sums approaches a fixed value; for example, $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots$ gets closer and closer to 1 as more and more terms are added.

- **divergent series**—A series with an infinite number of terms in which the sequence of partial sums does not approach a fixed value; for example, 2 + 4 + 8 + 16 + ... gets larger and larger as more terms are added.

A discussion of $S_n = \frac{t_1(1-r^n)}{1-r}; r \neq 1$ as *n* gets larger will lead students to an understanding of when geometric series converge and when they diverge. The formula $S_\infty = \frac{t_1}{1-r}$ will develop naturally from this discussion. Students will then apply this knowledge to contextual situations where they wish to sum an infinite geometric series where $-1 < r < 1$.

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

ASSESSING PRIOR KNOWLEDGE

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- What value of *x* would make the following statements true?

  (a) $5^x = 25$ (b) $2^{x-1} = 16$ (c) $3^x = \frac{1}{81}$

- A science experiment shows that the numbers of bacteria in a petri dish will double every hour. If there are 1000 bacteria after 8 hours, how many will there be after 10 hours?

- Evaluate:

  (a) $\frac{5x^{12}}{x^7}$ (b) $\frac{5^{110}}{5^{108}}$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Using pieces of square paper of varying side lengths, *n*, fold the paper in half repeatedly. After each fold, measure and record the area of the top surface. What type of sequence is formed? Explain your reasoning.

- Determine the first three terms of each of the following sequences: $t_n = 3n$ and $t_n = 2(3^n - 1)$. What do you notice about the value of t_1, t_2, and t_3 and the type of sequences created? Explain your reasoning.

- The formula for a geometric sequence is given by $t_n = t_1(r)^{n-1}$. Create examples of various geometric sequences in this form and explain what happens if

  (a) $r = 1$
  (b) $r > 1$
  (c) $0 < r < 1$
  (d) $r = 0$
  (e) $-1 > r > 0$
  (f) $r = -1$

- You are hired to complete a job for a month that offers two different payment options. In plan A, the payment begins with \$3 on day 1, \$6 on day 2, \$9 on day 3, etc. In plan B, the payment begins with \$0.01 on day 1, \$0.02 on day 2, \$0.04 on day 3, etc. Determine which payment plan would be more feasible. Explain why an employer might or might not offer a payment such as plan B.

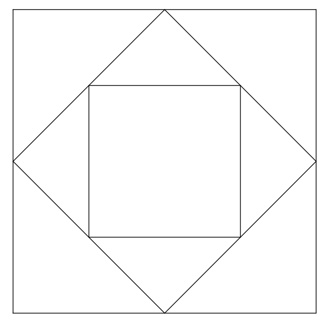
- A classmate is having trouble with the formula $t_n = t_1(r)^{n-1}$. He thinks the formula should be $t_n = t_1(r)^n$. How would you explain to him that he should use r^{n-1} rather than r^n in the formula?

- Explain why there can be no infinite geometric series with a first term of 12 and a sum of 5.

- Explain why the sum of an infinite geometric series is positive if, and only if, the first term is positive.

- Determine whether the following sequences converge or diverge. Explain your reasoning.
  (a) 8, 4, 2, 1, 0.5, ...
  (b) $3, \frac{7}{3}, \frac{5}{3}, 1, \frac{1}{3}, ...$
  (c) $5^{-3}, 5^{-2}, 5^{-1}, 5^{0}, ...$
  (d) $t_1 + d, t_1 + 2d, t_1 + 3d, t_1 + 4d, ...$

- The midpoints of a square with sides 1 m long are joined to form another square. Then the midpoints of the sides of the second square are joined to form a third square. This process is continued indefinitely to form an infinite set of smaller and smaller squares converging on the centre of the original square.
  (a) Determine the total length of the segments forming the sides of all the squares.
  (b) Determine the sum of the area of all the squares.



- Find t^5 for the geometric sequence $66, 22, \frac{22}{3}, ...$

- Determine the common ratio, and find the next three terms of the sequence
  (a) $2, 1, \frac{1}{2}, \frac{1}{4}, ...$
  (b) $2, -1, \frac{1}{2}, -\frac{1}{4}, ...$
  (c) $2, 2\sqrt{2}, 4, 4\sqrt{2}, ...$

- For the geometric sequence 5, 10, 20, 40, 80, ..., write an expression for the general term. Use it to find the value of t_8.

- A geometric sequence has 10 as the 3rd term and 80 as the 6th term. Find the value of the 8th term.

- Find the 11th term in the geometric sequence 256, –179.2, 125.44, ... Round off the answer to two decimal places.

- Write a geometric sequence that has 48 as the first term, –750 as the last term, and two other terms in between these two terms.

- After each washing, 1% of the remaining dye in a pair of blue jeans is washed out. What percentage of the dye is washed out after 10 washings? Round off the answer to two decimal places.

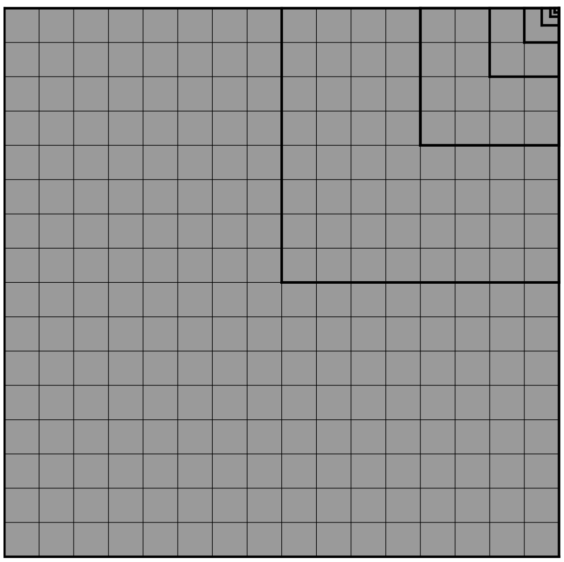
- Two years after its purchase, the resale value of a car was \$10, 000. The resale value of the same car three years later was \$5000. If the annual depreciation of the car forms a geometric sequence, what was the original price of the car? Round off the answer to the nearest dollar.

- Find S_8 for the geometric series 10 + 15 + 22.5 + ... Round off the answer to two decimal places.

- Find S_5 for the geometric series $\frac{1}{3}-\frac{2}{3}+\frac{4}{3}-\frac{8}{3}+\ldots$ Give the answer in exact form.

- Find the sum of the first 12 terms of the geometric series $1+\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+\ldots$ Round off the answer to two decimal places.

- What infinite geometric series is represented by the following diagram?



- Use an area model to illustrate that the sum of the infinite series $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\ldots$ is 1.

- Find the sum of the first ten terms of the geometric series 16 – 48 + 144 – 432 + ...
- A hockey tournament has 64 teams entered. When a team loses a game, it is eliminated. Winners play in the next round. How many games are played before the ultimate winner is determined?

- The tallest totem pole carved from a single log is 38.28 m high and is in Beacon Hill Park in Victoria, British Columbia. A lacrosse ball is dropped from this height and bounces back up to 60% of the original height. Find the total vertical distance travelled by the ball by the time it hits the ground for the tenth time. Round off the answer to one decimal place.

- A boy was hired to tend the king’s horses for 30 days. The king gave the boy two options for payment.

- **Option A**—One cent on the first day, two cents on the second day, and double his salary every day thereafter for the thirty days.
- **Option B**—Exactly \$1, 000, 000.

Which option was better and why?

- Determine the sum of each of the infinite geometric series, if it exists.
  (a) $t_1 = 10, r = \frac{1}{2}$
  (b) $t_1 = 4, r = -5$
  (c) $t_1 = -3, r = -\frac{1}{3}$

- Determine the sum of each of the infinite geometric series, if it exists.
  (a) $21 - 3 + \frac{3}{7} - ...$
  (b) $8 + 7 + \frac{49}{8} + ...$
  (c) 2 + 2.2 + 2.42 + ...

- Write each of the following repeating decimals as a fraction in lowest terms.
  (a) $0.\overline{21}$
  (b) $0.\overline{762}$

- In its first month, a well produced 15 000 m^3 of oil. Its production is known to be dropping by 2.9% each month. If the well is worked until it is dry, how much oil will be produced by the well in total? Round off the answer to the nearest thousand cubic metres.

- A super ball is dropped from a height of 40 m, and rebounds to 80% of its original height. Find the total vertical distance travelled by the ball before coming to rest.

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?

- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Students should provide their own examples of geometric sequences and illustrate the concept of a common ratio by determining the quotient of pairs of consecutive terms, $r = \frac{t_2}{t_1} = \frac{t_3}{t_2} = \frac{t_4}{t_3}$, demonstrating why any such sequence is geometric.

- It is important for students to think about what makes a sequence geometric as opposed to arithmetic. They should differentiate between a common difference and a common ratio. To develop the formula for a geometric sequence, ask students to consider several sequences such as 1, 3, 9, 27, ... and 1, –2, 4, –8, 16, ...

- Use the following questions to guide them through the process:
  - What is the common ratio?
  - Can this sequence be rewritten to show the pattern of the first term and the common ratio?
  - Can you predict the formula of a geometric sequence based on the patterns
  - $1, 1(3)^1, 1(3)^2, 1(3)^3, 1(3)^4, \ldots$ and $1, 1(-2)^1, 1(-2)^2, 1(-2)^3, 1(-2)^4$?
  - Can you write the pattern in general terms?

  Students should notice that the terms of the geometric sequence with first term t_1 and a common ratio r can be written in general as $t_1 = t_1$, $t_2 = t_1(r)^1$, $t_3 = t_1(r)^2$, $t_4 = t_1(r)^3$ ... They should recognize the formula for a geometric sequence is $t_n = t_1(r)^{n-1}$.

  Ask students to write the general term for sequences such as 600, 300, 150, 75, ... and 3, –6, 12, –24, ...

- Provide numerical and algebraic examples for students to evaluate and/or manipulate a given geometric sequence to determine the first term, the common ratio, the number of terms in the finite sequence, or the general rule.

- Encourage students to test the geometric series formula using the following numerical example. This would give them an opportunity to review geometric sequences while being introduced to geometric series.
  - A student is constructing a family tree. She is hoping to trace back through 10 generations to calculate the total number of ancestors she has. Determine the total number of ancestors after the 10th generation.

  Students should understand every person has two parents, four grandparents, eight great-grandparents, and so on. Therefore, the number of ancestors through ten generations is

2 + 4 + 8 + 16 + 32 + 64 + 128 + 256 + 512 + 1024. Let S represent the sum of this series. Ask students to multiply it by the common ratio 2.

$$S_{10} = 2+4+8+16+32+64+128+256+512+1024$$
$$2S_{10} = 4+8+16+32+64+128+256+512+1024+2048$$
$$2S_{10} - 1S_{10} = 2048 - 2$$
$$S_{10} = 2046$$

Students calculate the sum of the first ten terms of the series to be 2046. Going back through the generations, each person has 2046 ancestors.

- To find the sum of an infinite geometric series, students will first need to distinguish between a convergent and divergent series. To develop the idea of convergence or divergence, expose students to a variety of geometric sequences where the common ratio is different for each. Ask students to determine the partial sum by adding the first two terms, the first three terms and so on. They will check to see if the partial sum approaches a particular value as the number of terms get larger. Students are not familiar with limit notation nor is it an expectation for this course, but it could very easily be introduced here.

Geometric Series	**Partial Sum**	**Convergent/Divergent**
2 + 4 + 8 + 16 + ... ($r > 1$)	$S_1 = 2$, $S_2 = 6$, $S_3 = 14$	diverges
$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...$ ($-1 < r < 1$)	$S_1 = \frac{1}{2}, S_2 = \frac{3}{4}, S_3 = \frac{7}{8}$	converges to 1
–1 + 1 + –1 + 1 + ... ($r = -1$)	$S_1 = -1$, $S_2 = 0$, $S_3 = -1$, $S_4 = 0$	diverges
1 + 1 + 1 + 1 + ... ($r = 1$)	$S_1 = 1$, $S_2 = 2$, $S_3 = 3$, $S_4 = 4$	diverges

Once students determine the common ratio for each geometric series, they should state their conclusions about convergence or divergence based on the value of r. Ask students why the value of r cannot equal 0.

- Students could use a concrete representation to help with their understanding of an infinite geometric series. Using a string, 1 metre long, students will cut it in half and place one of the halves stretched out on a table. Using the remaining half, they will then cut it in half so that they have two quarters. Place one of the quarters at the end of the $\frac{1}{2}$ string on the table. They now have $\frac{3}{4}$ of the string on the table. Halve the remaining quarter string so they have two eighths and place one of the eighths at the end of the string on the table. They now have $\frac{7}{8}$ of the string on the table, and so on. The goal is for students to recognize that this infinite series represents a sequence of partial sums. Students can verify, using the formula $S_\infty = \frac{t_1}{1-r}$ that the sum should equal 1.

- Remind students that the value of n must be a natural number; that is, n can only take on the values 1, 2, 3, 4, ...

- Ensure that students understand, that for positive values of *r*, if the value of *r* is greater than one, then the geometric sequence is increasing, and that if the value of *r* is less than one, then the geometric sequence is decreasing.

- Derive the formula for the sum of the first *n* terms of a general geometric series with the students.

- Develop the formula for the sum of an infinite geometric series, when $-1 < r < 1$ and demonstrate that the formula for the sum of an infinite geometric series does not work for a divergent series, such as 1 + 2 + 4 + 8 + ...

- The idea of limits can be very confusing. It is important that students understand that if a series converges on 2, as you add up more and more of the terms, the sum of that finite series is getting closer and close to 2 *and* that the actual sum of the infinite series is 2. Many students believe that a convergent infinite geometric series approaches a sum, as opposed to being equal to a sum. Ensure that they understand that the sum is in fact the sum of the entire infinite geometric series.

  Demonstrate to students that 0.999 ... = 1, by writing 0.999 ... as an infinite series of 0.999 ... = 0.9 + 0.09 + 0.009 + ... and then finding its sum.

  $$t_1 = 0.9$$
  $$r = \frac{0.09}{0.9} = \frac{1}{10} = 0.1$$
  $$S_\infty = \frac{t_1}{1-r} = \frac{0.9}{1-0.1} = \frac{0.9}{0.9} = 1$$

  You may also wish to use the following approach to further verify this fact.

  $$\text{Let } x = 0.999...$$
  $$\therefore 10x = 9.999...$$
  $$10x - x = 9.999... - 0.999...$$
  $$9x = 9$$
  $$\therefore x = 1$$

- When working through the problems, remind students of the following guidelines:
  - Draw a diagram if necessary.
  - Write out the terms of the sequence.
  - Make sure that you understand exactly what each term is representing.
  - Choose to let t_1 represent the term that is most easily associated with the number 1 even if it is not the first term in the sequence you wrote out.
  - Determine if the problem requires you to use a sequence or series.
  - Determine if the problem is arithmetic or geometric.
  - Construct the formula using the given information in the problem.
  - Solve the problem.

  For example:
  - After each washing, 1% of the remaining dye in a pair of blue jeans is washed out. What percentage of the dye is washed out after 10 washings?
  - 100%, 99%, 98.01%, 97.03%, ...
    - > 100% is the amount before the jeans are washed.

> 99% is the amount of dye remaining after 1 wash ... so it is most convenient to chose for t_1 to be 99% since then t_n will represent the amount of dye remaining after n washes.

$$t_1 = 99\%$$
$$r = \frac{99}{100} = \frac{98.01}{99} = 0.99$$
$$n = 10$$

$$t_n = t_1(r)^{n-1}$$
$$t_{10} = 99\%(0.99)^{10-1}$$
$$t_{10} = 99\%(0.99)^{9} = 90.44\%$$

> After 10 washes there would still be 90.44% of the dye in the jeans.

- Be careful to only assign questions that students can solve without the use of logarithms. At this point all questions must be able to be solved by equating bases. (Logarithms and Exponential functions and equations are taught in detail in Pre-calculus 12).

  For example, it would not be possible for students to solve the following question except by graphing.
  - After each washing, 1% of the remaining dye in a pair of blue jeans is washed out. How many washings would be necessary to reduce the amount of dye to 80%?

  However, it would be possible for students to determine how many terms there are in the sequence 6, 12, 24, 48, ..., 1536, since 1536 is the exact value of a specific term; therefore, it is possible to solve this question by equating the bases.

$$t_1 = 6$$
$$r = \frac{12}{6} = \frac{24}{12} = 2$$
$$n = ?$$
$$t_n = 1536$$

$$t_n = t_1(r)^{n-1}$$
$$1536 = 6(2)^{n-1}$$
$$256 = (2)^{n-1}$$
$$2^8 = 2^{n-1}$$
$$8 = n-1$$
$$n = 9$$

SUGGESTED MODELS AND MANIPULATIVES

- coins
- counting disks
- Cube-A-Links
- graph paper
- ruler

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- common ratio
- convergent series
- divergent series
- explicit notation
- general term
- geometric sequence
- geometric series
- infinite series
- recursive notation

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Sections 1.3, 1.4, and 1.5, pp. 32–65

Notes

SCO RF11 Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions). [CN, R, T, V]			
[C] Communication	**[PS]** Problem Solving	**[CN]** Connections	**[ME]** Mental Mathematics and Estimation
[T] Technology	**[V]** Visualization	**[R]** Reasoning	

Performance Indicators

Use the following set of indicators to determine whether students have achieved the corresponding specific curriculum outcome.

RF11.01 Compare the graph of $y=\dfrac{1}{f(x)}$ to the graph of $y = f(x)$.

RF11.02 Identify, given a function $f(x)$, values of x for which $y=\dfrac{1}{f(x)}$ will have vertical asymptotes; and describe their relationship to the non-permissible values of the related rational expression.

RF11.03 Graph, with or without technology, $y=\dfrac{1}{f(x)}$, given $y = f(x)$ as a function or a graph, and explain the strategies used.

RF11.04 Graph, with or without technology, $y = f(x)$, given $y=\dfrac{1}{f(x)}$ as a function or a graph, and explain the strategies used.

Scope and Sequence

Mathematics 10 / Mathematics 11	**Pre-calculus 11**	**Pre-calculus 12**
RF05 Students will be expected to determine the characteristics of the graphs of linear relations, including the intercepts, slope, domain, and range. (M10)* **RF06** Students will be expected to relate linear relations expressed in ▪ slope-intercept form ($y = mx + b$) ▪ general form ($Ax + By + C = 0$) ▪ slope-point form ($y - y_1) = m(x - x_1$) (M10) **RF02** Students will be expected to demonstrate an understanding of the characteristics of quadratic functions, including vertex, intercepts, domain and range, and axis of symmetry. (M11)**	**RF11** Students will be expected to graph and analyze reciprocal functions (limited to the reciprocal of linear and quadratic functions).	**RF14** Students will be expected to graph and analyze rational functions (limited to numerators and denominators that are monomials, binomials, or trinomials). **T03** Students will be expected to solve problems, using the six trigonometric ratios for angles expressed in radians and degrees.

* M10—Mathematics 10
** M11—Mathematics 11

Background

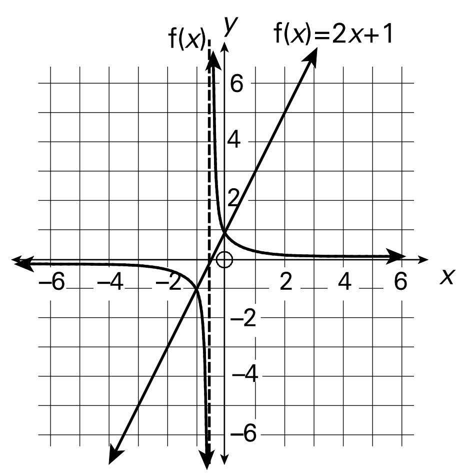
In Mathematics 10, students graphed linear functions and in Mathematics 11, students graphed quadratic functions. In this course, students will compare the graphs of a function and its reciprocal. Examples will include both linear and quadratic functions.

When graphing a function and its reciprocal, there are points that remain unchanged. These are called **invariant points**. Exploring points on a function such as $f(x) = 2x + 1$ and its reciprocal $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$ will have students explore the idea that the invariant points can be located by setting $f(x) = \pm 1$.

x	$f(x)$	$\frac{1}{f(x)}$
1	3	$\frac{1}{3}$
0.5	2	$\frac{1}{2}$
0	1	1
–0.5	0	undefined
–1	–1	–1

These points will help students when they graph reciprocal functions, as these will be common to both functions and are, therefore, the points of intersection.

Ask students to graph $f(x) = 2x + 1$ and its reciprocal $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$ by plotting the points that they obtained. If they are not certain how to join the points for the reciprocal function, then ask them to plot additional points until they feel confident enough to connect their plotted points.



You may wish to ask students questions such as,

- as $x \to -\frac{1}{2}$ from the right, what is happening to $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$?
- as $x \to -\frac{1}{2}$ from the left, what is happening to $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$?
- when $x = \frac{1}{2}$, what is true about the function $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$?
- what is happening to $g(x) = \frac{1}{f(x)} = \frac{1}{2x+1}$ when $x \to \pm\infty$?

Similarly this exploration will lead students to a discussion of when the reciprocal of a function is undefined. Students will conclude that when $f(x) = 0$ the reciprocal function has a **vertical asymptote** and that the reciprocal function has a **horizontal asymptote** of $y = 0$.

As students explore linear and quadratic reciprocal functions, there are certain characteristics that should be addressed in a class discussion.

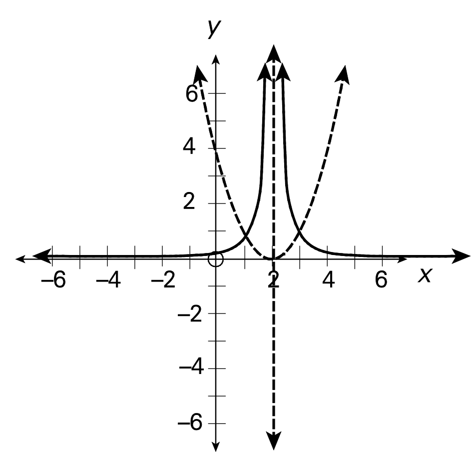
Students should consider the following:

- If the point (x, y) is on the graph $y = f(x)$, what do you notice about the point on the graph of the reciprocal function?
- Is the sign of the reciprocal function the same as the sign of the original function? In other words, when the original function is positive, is the reciprocal function positive or negative? When the original function is negative, is the reciprocal function positive or negative?
- What do you notice about the x-intercept of the linear function, the non-permissible value of the reciprocal function, and the location of the vertical asymptote?
- What is the horizontal asymptote?
- What are the invariant points?
- Why do the intervals of decrease on the original function become intervals of increase on the reciprocal function or remain intervals of decrease?
- Why do the intervals of increase on the original function become intervals of decrease on the reciprocal function or remain intervals of increase?
- What do you notice about the behaviour of the reciprocal function as it approaches the asymptotes?

They should also recognize, from the table and the graph, that the x-intercept of the function produces a non-permissible value for the reciprocal function. This is the location of the vertical asymptote. When analyzing and comparing the graphs of the function and its reciprocal, it is important to note that as the y-values of one function increase the y-values of the other decrease and vice versa. This could be demonstrated by looking at parts of the graph on either side of the invariant points.

As students recognize the shape of the graph along with the intercepts and the asymptotes, they should be able to sketch the graph of the reciprocal function.

Once students have explored the graphs of linear functions and their reciprocals, they will extend their exploration to the graphs of quadratic functions and their reciprocals. There are three cases that need to be considered and explored with students, namely, quadratic functions that have one, two, or no x-intercepts.



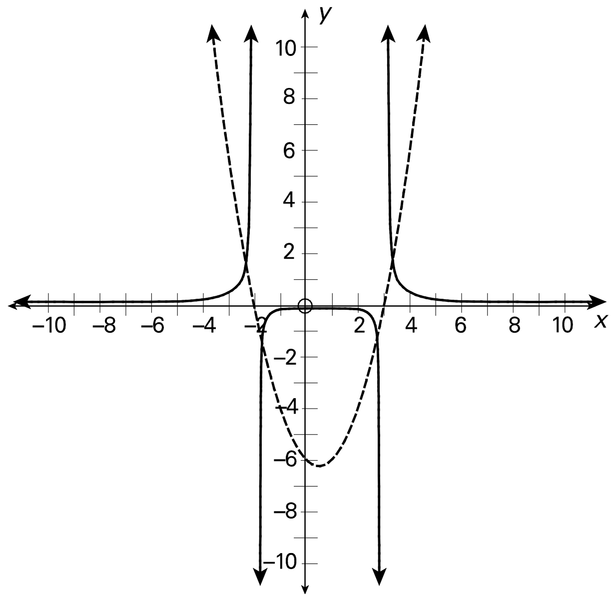
First investigate a quadratic function with one x-intercept.

Consider the graph of $y = (x - 2)^2$ and its reciprocal $y = \dfrac{1}{(x-2)^2}$.

Using the graph as a visual aid, students should observe the following:

- There is one x-intercept on the graph of $y = f(x)$ located at $x = 2$. This corresponds to the location of the vertical asymptote on the graph of $y = \dfrac{1}{f(x)}$.
- The graphs of $y = f(x)$ and $y = \dfrac{1}{f(x)}$ intersect where $f(x) = \pm 1$ (in this case only at $f(x) = 1$). These are the invariant points.

- The graph of $y=\dfrac{1}{f(x)}$ is also asymptotic to the x-axis. The horizontal asymptote is located at $y = 0$.



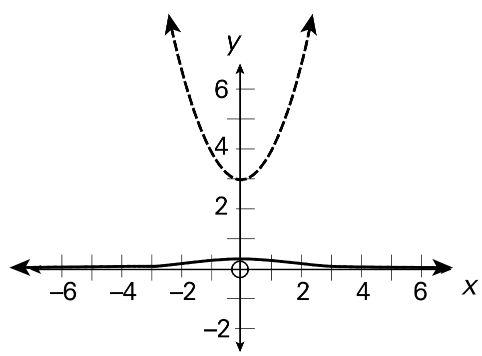
Similarly, students can investigate a quadratic function with two x-intercepts. Consider the graph of $y = (x - 3)(x + 2)$ and its reciprocal $y=\dfrac{1}{(x-3)(x+2)}$.

Students should note the following features:

- The x-intercepts of the graph of $y = f(x)$, $x = -2$, and $x = 3$ correspond to the location of the vertical asymptotes on the graph of $y=\dfrac{1}{f(x)}$.
- The graphs of $y = f(x)$ and $y=\dfrac{1}{f(x)}$ intersect at $f(x)=\pm1$. These are the invariant points.
- The equation of the horizontal asymptote is located at $y = 0$.
- The parts of the graph of $y = f(x)$ that were positive remain positive on the graph of $y=\dfrac{1}{f(x)}$. The parts of the graph of $y = f(x)$ that were negative remain negative on the graph of $y=\dfrac{1}{f(x)}$.

When graphing a quadratic function with no x-intercepts and its reciprocal, students should notice that the characteristics are different from the previous cases.

Consider the graph of $y = x^2 + 3$ and $y=\dfrac{1}{x^2+3}$.



- The graph of $y = f(x)$ has no x-intercepts. Therefore, the graph of $y=\dfrac{1}{f(x)}$ has no vertical asymptotes.
- The graph of $y = f(x)$ does not have any points at $f(x)=\pm1$. Therefore, there are no points of intersection between the graphs (i.e., there are no invariant points).
- The equation of the horizontal asymptote is located at $y = 0$.

This investigation helps students recognize the shape of the graph of a reciprocal function. Using the intercepts and asymptotes, students can proceed to graph a reciprocal function.

Students will also work backwards and sketch the graph of the original function given the graph of the reciprocal function. They should be able to determine by inspection whether the original function is linear or quadratic by analyzing the reciprocal function. It is important to remind students of the key concepts when working with reciprocal functions.

- The vertical asymptote of the reciprocal function is the x-intercept(s) of the original graph.
- The point (x, y) on the reciprocal function becomes $\left(x, \frac{1}{y}\right)$ on the original function.
- The two functions will intersect when $f(x) = \pm 1$.

(**Note:** In Pre-calculus 12, students will explore the graphs of $y = \frac{1}{x}$ and $y = \frac{1}{x^2}$ and transformations of these functions.)

Assessment, Teaching, and Learning

Assessment Strategies

Assessment for learning can and should happen most days as a part of instruction. Assessment of learning should also occur frequently. A variety of approaches and contexts could be used for assessing the learning of all students—as a class, in groups, and individually.

Guiding Questions

- What are the most appropriate methods and activities for assessing student learning?
- How will I align my assessment strategies with my teaching strategies?

Assessing Prior Knowledge

Student tasks such as the following could be completed to assist in determining students' prior knowledge.

- Graph the following functions.
  (a) $f(x) = 2x - 6$
  (b) $g(x) = (x + 1)(x - 3)$
  (c) $h(x) = -2(x - 1)^2 + 4$
  (d) $m(x) = x^2 + 1$

WHOLE-CLASS/GROUP/INDIVIDUAL ASSESSMENT TASKS

Consider the following sample student tasks and activities (that can be adapted) as either assessment for learning or assessment of learning.

- Describe the reasoning you can use to decide whether each statement in the chart below is true or false.

Consider the function $f(x)=\frac{1}{3x-4}$.	True	False	Why I (we) think so.
▪ There is a vertical asymptote at $x=\frac{4}{3}$.			
▪ There is a horizontal asymptote at $y = 0$.			
▪ $f(0)=\frac{1}{4}$			
▪ The point (4, 8) is on the graph of $y = 3x - 4$ and the point (4, –8) is on the graph of $y=\frac{1}{3x-4}$.			

- Describe the reasoning you can use to decide whether each statement in the chart below is true or false.

Consider the function $f(x)=\frac{1}{x^2-5x+6}$.	True	False	Why I (we) think so.
▪ There is a vertical asymptote at $x = 2$.			
▪ There is a vertical asymptote at $x = -3$.			
▪ $f(0)=\frac{1}{6}$			
▪ $y=\frac{1}{f(x)}$ has x-intercepts at 2 and 3.			

- Explain how you could graph the function $k(x)=\frac{1}{(x+1)(x-5)}$.

- Explain how you could graph the reciprocal of a quadratic function that opens down and has a vertex of (3, 0).

- Explain how you could graph the reciprocal of a quadratic function with no x-intercepts.

- State and then graph the reciprocal function for each of the following:
  (a) $f(x) = 8 - 2x$
  (b) $g(x) = 4x + 6$
  (c) $h(x) = 4x^2 + 4x + 1$
  (d) $m(x) = -1(x - 2)^2 + 3$
  (e) $k(x) = 2(x + 1)(x - 4)$

- State and then graph the reciprocal function for each of the following:
  (a) $f(x) = \dfrac{1}{4 - 8x}$
  (b) $g(x) = \dfrac{1}{x^2 - 9}$
  (c) $h(x) = \dfrac{1}{(x - 3)^2}$
  (d) $m(x) = \dfrac{1}{x^2 + 9}$

- Graph the reciprocal function for each of the following:

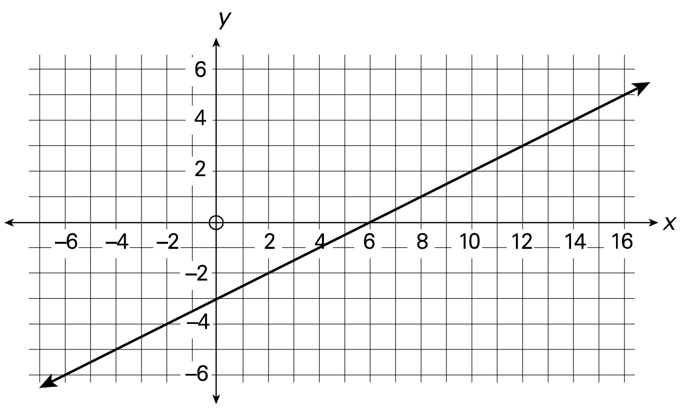


Figure 9

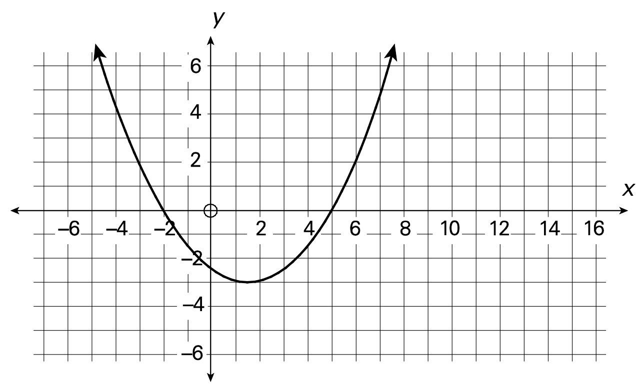


Figure 10

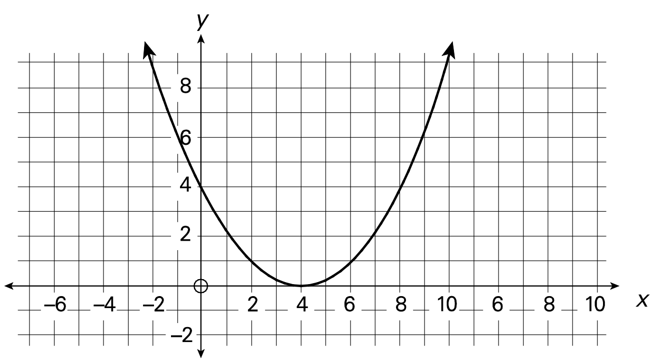


Figure 11

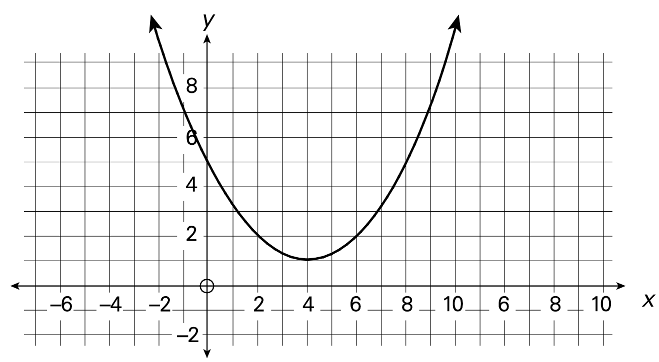


Figure 12

- Graph the reciprocal function for each of the following:

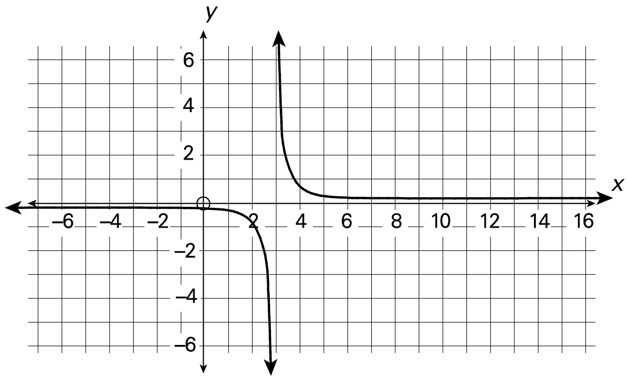


Figure 13

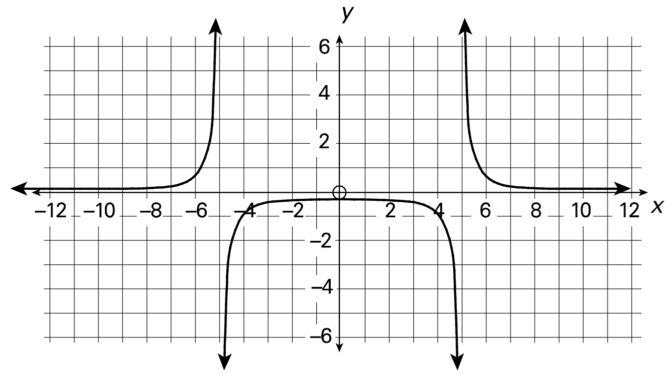


Figure 14

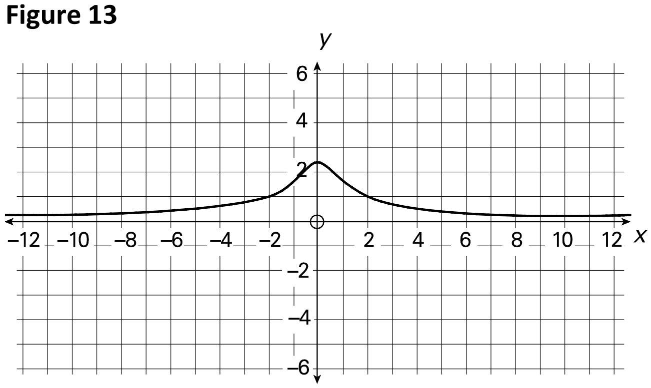


Figure 15

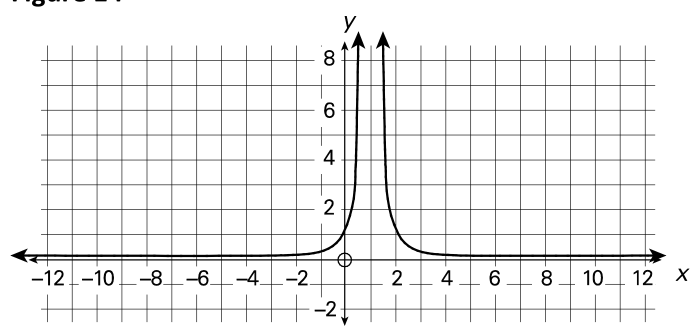


Figure 16

FOLLOW-UP ON ASSESSMENT

The evidence of learning that is gathered from student participation and work should inform instruction.

Guiding Questions
- What conclusions can be made from assessment information?
- How effective have instructional approaches been?
- What are the next steps in instruction for the class and for individual students?
- What are some ways students can be given feedback in a timely fashion?

Planning for Instruction

Planning for a coherent instructional flow is a necessary part of an effective mathematics program.

Guiding Questions
- Does the lesson fit into my yearly/unit plan?
- How can the processes indicated for this outcome be incorporated into instruction?
- What learning opportunities and experiences should be provided to promote achievement of the outcomes and permit students to demonstrate their learning?
- What teaching strategies and resources should be used?
- How will the diverse learning needs of students be met?
- How can the scope and sequence be used to determine what prior knowledge needs to be activated prior to beginning new instruction?

SUGGESTED LEARNING TASKS

Consider the following sample instructional strategies when planning lessons.

- Ask students to consider $\frac{1}{n}$.
  - When $n = 3$, what is $\frac{1}{n}$?
  - When $n = 30$, what is $\frac{1}{n}$?
  - When $n = 1000$, what is $\frac{1}{n}$?
  - For a very large value of n, what is true of $\frac{1}{n}$?
  - For a very small value of n, what is true of $\frac{1}{n}$?

- Students may forget to use brackets when putting the reciprocal function into a graphing calculator or other graphing utility. Thus, they may graph $y=\frac{1}{x}-4$ when they intend to graph $y=\frac{1}{x-4}$.

- Ask students to use graphing technology to analyze and describe the characteristics of a function and its reciprocal, and vice versa.

- Challenge students to consider all types of both linear and quadratic functions. They need to consider various possibilities such as linear functions with a positive slope and those with a negative slope as well as quadratic functions that open up and those that open down. They need to explore quadratic functions that have two *x*-intercepts, one *x*-intercept, and no *x*-intercepts. These explorations take time. The use of a graphing program such as Autograph is preferable to the use of a graphing calculator since the graphing calculator sometimes joins points that should not be joined and creates a graph that is not accurate.

- As teachers observe this task, they should ask questions such as
  - Where do the graphs of a function and its reciprocal intersect?
  - What do the vertical asymptotes of the reciprocal function correspond to in the original function?
  - What is happening to the reciprocal function as the values of *x* approach positive infinity? As they approach negative infinity?
  - When the original function is positive what is the sign of the reciprocal function? When the original function is negative what is the sign of the reciprocal function?

- Provide students with a reciprocal function. Ask them to sketch the graph of the original function, $y = f(x)$, and explain the strategies they used.

- Some possible activities that reinforce the reciprocal functions are described below.

  - Students can work in pairs for this task. Give one student the graph of a quadratic function or its reciprocal on the same set of axes. Ask him/her to turn to their partner and describe, using the characteristics of the quadratic function or its reciprocal, the graph they see. The other student will draw the graph based on the description from the student. Both students will then check to see if their graphs match.

- Working in small groups, play the Reciprocal Matching Game. Each group will be given several cards. Half of the cards will have the graph of $y = f(x)$, while the other half will have the graph of $y = \frac{1}{f(x)}$. The object of the game is to be the first group to pair up each graph with its reciprocal graph.

SUGGESTED MODELS AND MANIPULATIVES

- graph paper
- ruler

MATHEMATICAL VOCABULARY

Students need to be comfortable using the following vocabulary.

- asymptote
- horizontal asymptote
- invariant points
- reciprocal function
- vertical asymptote

Resources/Notes

Print

- *Pre-Calculus 11* (McAskill et al. 2011)
  - Section 7.4, pp. 392–409

Notes

References

Alberta Education. 2003. "Exploring Laws of Exponents—Use It, " *Math Interactives*. Alberta Education. (www.learnalberta.ca/content/mejhm/index.html?l=0&ID1=AB.MATH.JR.NUMB&ID2=AB.MATH.JR.NUMB.EXPO&lesson=html/object_interactives/exponent_laws/use_it.html)

American Association for the Advancement of Science [AAAS-Benchmarks]. 1993. *Benchmark for Science Literacy*. New York, NY: Oxford University Press.

Armstrong, T. 1993. *Seven Kinds of Smart: Identifying and Developing Your Many Intelligences*. New York, NY: Plume.

Black, Paul, and Dylan Wiliam. 1998. "Inside the Black Box: Raising Standards Through Classroom Assessment." *Phi Delta Kappan 80*, No. 2 (October 1998), pp. 139–144, 146–148.

British Columbia Ministry of Education. 2000. *The Primary Program: A Framework for Teaching*. Victoria, BC: Province of British Columbia.

Caine, Renate Numella, and Geoffrey Caine. 1991. *Making Connections: Teaching and the Human* Brain. Reston, VA: Association for Supervision and Curriculum Development.

Davies, Anne. 2000. *Making Classroom Assessment Work*. Courtenay, BC: Classroom Connections International, Inc.

Frankenstein, Marilyn. 1995. "Equity in Mathematics Education: Class in the World outside the Class." *New Directions for Equity in Mathematics Education*. Cambridge, MA: Cambridge University Press.

Gardner, Howard E. 2007. *Frames of Mind: The Theory of Multiple Intelligences*. New York, NY: Basic Books.

Gutstein, Eric. 2003. "Teaching and Learning Mathematics for Social Justice in an Urban, Latino School." *Journal for Research in Mathematics Education 34*, No. 1. Reston, VA: National Council of Teachers of Mathematics.

Hall, Tracey E., Anne Meyer, and David H. Rose (ed.). 2012. *Universal Design for Learning in the Classroom: Practical Applications*. New York, NY: The Guilford Press.

Herzig, Abbe. 2005. "Connecting Research to Teaching: Goals for Achieving Diversity in Mathematics Classrooms." *Mathematics Teacher*, Volume 99, No. 4. Reston, VA: National Council of Teachers of Mathematics.

Hope, Jack A., Larry Leutzinger, Barbara Reys, and Robert Reys. 1988. *Mental Math in the Primary Grades*. Palo Alto, CA: Dale Seymour Publications.

Hume, Karen. 2011. *Tuned Out: Engaging the 21st Century Learner*. Don Mills, ON: Pearson Education Canada.

Ladson-Billings, Gloria. 1997. "It Doesn't Add Up: African American Students' Mathematics Achievement." *Journal for Research in Mathematics Education 28*, No. 6. Reston, VA: National Council of Teachers of Mathematics.

McAskill, Bruce, Wayne Watt, Eric Balzarini, Len Bonifacio, Scott Carlson, Blaise Johnson, Ron Kennedy, and Harold Wardrop. 2011. *Pre-Calculus 11*. Whitby, ON: McGraw-Hill Ryerson.

National Council of Teachers of Mathematics. 2000. *Principles and Standards for School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

———. 2005. “Computation, Calculators, and Common Sense: A Position of the National Council of Teachers of Mathematics” (position paper, May 2005). Reston, VA: National Council of Teachers of Mathematics.

New Brunswick Department of Education and Early Childhood Development. 2014. *Pre-Calculus 110 Curriculum*. Fredericton, NB: Province of New Brunswick.

Newfoundland and Labrador Department of Education. 2012. *Mathematics: Academic Mathematics 2200*, Interim Edition. St. John’s, NL: Government of Newfoundland and Labrador.

———. 2014. “Quadratic Equations.” *Government of Newfoundland and Labrador.* www.k12pl.nl.ca/curr/10-12/math/math2201/classroomclips/quadeq.html

Nova Scotia Department of Education. 2002. *Racial Equity Policy*. Halifax, NS: Province of Nova Scotia.

———. 2010. *Gifted Education and Talent Development*. Halifax, NS: Province of Nova Scotia.

———. 2011. *Racial Equity / Cultural Proficiency Framework*. Halifax, NS: Province of Nova Scotia.

OECD Centre for Educational Research and Innovation. 2006. *Formative Assessment: Improving Learning in Secondary Classrooms*. Paris, France: Organization for Economic Co-operation and Development (OECD) Publishing.

Prince Edward Island Department of Education and Early Childhood Development. 2012. *Mathematics: MAT521B*. Summerside, PE: Government of Prince Edward Island.

Professional Learning K–12, Newfoundland and Labrador. 2013. *Professional Learning Newfoundland and Labrador* (www.k12pl.nl.ca)

Rubenstein, Rheta N. 2001. “Mental Mathematics beyond the Middle School: Why? What? How?” *Mathematics Teacher*, September 2001, Vol. 94, No. 6. Reston, VA: National Council of Teachers of Mathematics.

Shaw, J. M., and Cliatt, M. F. P. 1989. “Developing Measurement Sense.” In P.R. Trafton (Ed.), *New Directions for Elementary School Mathematics*. Reston, VA: National Council of Teachers of Mathematics.

Steen, L. A. (ed.). 1990. *On the Shoulders of Giants: New Approaches to Numeracy*. Washington, DC: National Research Council.

System Improvement Group, Alberta Education. 2006. *Western and Northern Canadian Protocol (WNCP) Consultation with Post-Secondary Institutions, Business and Industry Regarding Their Requirements for High School Mathematics: Final Report on Findings*, January 25, 2006. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.

Tate, William F. 1995. “Returning to the Root: A Culturally Relevant Approach to Mathematics Pedagogy.” *Theory into Practice 34*, Issue 3. Florence, KY: Taylor & Francis.

Tomlinson, Carol Ann. 2001. *How to Differentiate Instruction in Mixed-Ability Classrooms*. Alexandria, VA. Association for Supervision and Curriculum Development.

Western and Northern Canadian Protocol (WNCP) for Collaboration in Education. 2006. *The Common Curriculum Framework for K–9 Mathematics*. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.

———. 2008. *The Common Curriculum Framework for Grades 10–12 Mathematics*. Edmonton, AB: Western and Northern Canadian Protocol (WNCP) for Collaboration in Education.